

NON-COMMUTATIVE VIRTUAL STRUCTURE SHEAVES

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ABSTRACT. The moduli spaces of stable sheaves on projective schemes admit certain gluing data of Kapranov's NC structures, which we call quasi NC structures. The formal completion of the quasi NC structure at a closed point coincides with the pro-representable hull of the non-commutative deformation functor of the corresponding sheaf. In this paper, we show the existence of smooth non-commutative dg-resolutions of the above quasi NC structures, and call them quasi NCDG structures. When there are no higher obstruction spaces, the quasi NCDG structures define the notion of NC virtual structure sheaves, the non-commutative analogue of virtual structure sheaves. We show that the NC virtual structure sheaves are described in terms of usual virtual structure sheaves together with Schur complexes of the perfect obstruction theories.

1. INTRODUCTION

The purpose of this paper is to introduce the notion of *non-commutative virtual structure sheaves* on the moduli spaces of stable sheaves on projective schemes without higher obstruction spaces. The motivation introducing this concept is to construct non-commutative analogue of the enumerative invariants of sheaves, e.g. Donaldson-Thomas (DT) invariants [Tho00], involving non-commutative deformations of sheaves. In this introduction, we first recall some background of commutative virtual structure sheaves via smooth commutative dg-schemes, and also explain quasi NC structures on the moduli spaces of stable sheaves obtained in [Toddb]. We then state the existence of smooth non-commutative dg-enhancements on the moduli spaces of stable sheaves, called quasi NCDG structures, which govern the commutative dg-enhancements and the quasi NC structures. The construction of quasi NCDG structures leads to the definition of NC virtual structure sheaves.

1.1. Commutative virtual structure sheaves. Let X be a projective scheme, and M_α the moduli space of stable sheaves on X with Hilbert polynomial α . In general, the moduli space M_α may not have the expected dimension at $[E] \in M_\alpha$:

$$(1.1.1) \quad \exp.\dim_{[E]} M_\alpha := \dim \operatorname{Ext}^1(E, E) - \dim \operatorname{Ext}^2(E, E).$$

Here $\operatorname{Ext}^1(E, E)$ is the tangent space of M_α at $[E] \in M_\alpha$, and $\operatorname{Ext}^2(E, E)$ is the obstruction space. As long as there are no higher obstruction spaces, i.e.

$$(1.1.2) \quad \operatorname{Ext}^{\geq 3}(E, E) = 0 \quad \text{for any } [E] \in M_\alpha$$

the expected dimension (1.1.1) is locally constant on M_α , and in this case the virtual fundamental class $[M_\alpha]^{\text{vir}} \in A_*(M_\alpha)$ with the expected dimension (1.1.1) can be constructed using the notion of perfect obstruction theory [BF97]. The integration of the virtual fundamental class yields interesting enumerative invariants of sheaves, such as DT invariants [Tho00].

From the construction of the virtual class, it admits a natural K-theoretic enhancement (cf. [BF97, Remark 5.4])

$$(1.1.3) \quad \mathcal{O}_{M_\alpha}^{\text{vir}} \in K_0(M_\alpha)$$

called the *virtual structure sheaf* of M_α . It recovers the virtual fundamental class by applying the cycle map to (1.1.3). It was also suggested by Kontsevich [Kon95] that M_α may be obtained as the zero-th truncation of a smooth commutative dg-scheme $(N_\alpha, \mathcal{O}_{N_\alpha, \bullet})$, i.e. $\mathcal{O}_{M_\alpha} = \mathcal{H}_0(\mathcal{O}_{N_\alpha, \bullet})$, and the virtual structure sheaf (1.1.3) may be described as

$$(1.1.4) \quad \mathcal{O}_{M_\alpha}^{\text{vir}} = \sum_{i \in \mathbb{Z}} (-1)^i [\mathcal{H}_i(\mathcal{O}_{N_\alpha, \bullet})].$$

The dg-scheme $(N_\alpha, \mathcal{O}_{N_\alpha, \bullet})$ was later constructed by Toën-Vaquié [TV07] and Behrend-Fontanine-Hwang-Rose [BFHR14]. Also the identity (1.1.4) was established by Fontanine-Kapranov [CFK09].

1.2. Quasi NC structures on M_α . In the previous paper [Todb], the moduli space M_α turned out to admit a certain non-commutative structure, giving an enhancement of M_α different from the commutative dg-enhancement $(N_\alpha, \mathcal{O}_{N_\alpha, \bullet})$. Such a non-commutative structure was formulated in terms of Kapranov's NC schemes [Kap98], which are ringed spaces whose structure sheaves are possibly non-commutative, but formal in the non-commutative direction. We refer to [PT14], [Ore] for the recent developments on Kapranov's NC schemes.

The above non-commutative structure can be naturally observed from the formal deformation theory. For $[E] \in M_\alpha$, the formal deformation theory of E is governed by the dg-algebra $\mathbf{R}\text{Hom}(E, E)$, which is quasi-isomorphic to a minimal A_∞ -algebra

$$(1.2.1) \quad (\text{Ext}^*(E, E), \{m_n\}_{n \geq 2}).$$

The formal solution of the Maurer-Cartan equation of the A_∞ -algebra (1.2.1) yields the not necessary commutative algebra

$$(1.2.2) \quad R_E^{\text{nc}} := \frac{\widehat{T}(\text{Ext}^1(E, E)^\vee)}{\left(\sum_{n \geq 2} m_n^\vee\right)}.$$

Here m_n^\vee is the dual of the A_∞ -product

$$m_n: \text{Ext}^1(E, E)^{\otimes n} \rightarrow \text{Ext}^2(E, E).$$

The algebra (1.2.2) is an enhancement of the commutative algebra $\widehat{\mathcal{O}}_{M_\alpha, [E]}$ in the sense that

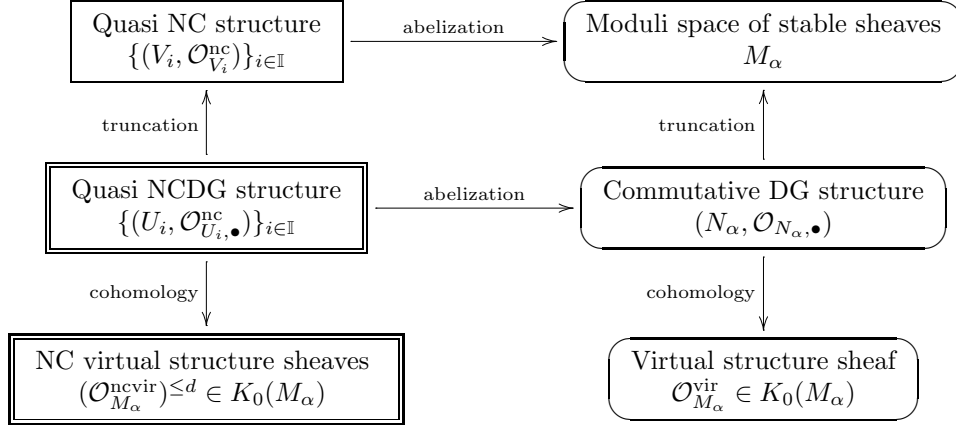
$$(1.2.3) \quad (R_E^{\text{nc}})^{ab} \cong \widehat{\mathcal{O}}_{M_\alpha, [E]}.$$

Indeed, the algebra R_E^{nc} is a pro-representable hull of the non-commutative deformation functor of E developed in [Lau02], [Eri10], [Seg08], [ELO09], [ELO10], [ELO11]. The main result of [Todb] was to construct a kind of globalization of the isomorphism (1.2.3), as follows:

Theorem 1.1. ([Todb, Theorem 1.2]) *There exists an affine open cover $\{V_i\}_{i \in \mathbb{I}}$ of M_α , ringed spaces V_i^{nc} and isomorphisms ϕ_{ij}*

$$(1.2.4) \quad V_i^{\text{nc}} = (V_i, \mathcal{O}_{V_i}^{\text{nc}}), \quad \phi_{ij}: V_j^{\text{nc}}|_{V_{ij}} \xrightarrow{\cong} V_i^{\text{nc}}|_{V_{ij}}$$

FIGURE 1. Relations of DG structures, quasi NC structures and quasi NCDG structures



where V_i^{nc} is Kapranov's affine NC scheme [Kap98], such that $\phi_{ij}^{ab} = \text{id}$ and $\hat{\mathcal{O}}_{V_i, [E]}^{\text{nc}} \cong R_E^{\text{nc}}$ for any $[E] \in V_i$.

The data (1.2.4) was called a *quasi NC structure* of M_α in [Toddb].

1.3. Quasi NCDG structures. Now we have obtained two kinds of enhancement of M_α : a commutative dg structure and a quasi NC structure. It is a natural question to construct a further enhancement which governs these two structures. We introduce the notion of *quasi NCDG structures* on commutative dg-schemes, and answer this question. Roughly speaking, a quasi NCDG structure on a commutative dg-scheme $(N, \mathcal{O}_{N, \bullet})$ is an affine open cover $\{U_i\}_{i \in \mathbb{I}}$ of N together with sheaves of non-commutative dg-algebras $\mathcal{O}_{U_i, \bullet}^{\text{nc}}$ on U_i satisfying the following:

- We have $(\mathcal{O}_{U_i, \bullet}^{\text{nc}})^{ab} = \mathcal{O}_{N, \bullet}|_{U_i}$.
- We have the isomorphisms $\phi_{ij, \bullet}: (U_{ij}, \mathcal{O}_{U_{ij}, \bullet}^{\text{nc}}|_{U_{ij}}) \xrightarrow{\cong} (U_{ij}, \mathcal{O}_{U_i, \bullet}^{\text{nc}}|_{U_{ij}})$ satisfying $\phi_{ij, \bullet}^{ab} = \text{id}$.

We will show the following result:

Theorem 1.2. (Theorem 5.4) *There is a smooth quasi NCDG structure on the dg-moduli space $(N_\alpha, \mathcal{O}_{N_\alpha, \bullet})$ whose zero-th truncation gives a quasi NC structure (1.2.4).*

The quasi NCDG structure in Theorem 1.2 fits into the upper half of the picture in Figure 1.3.

1.4. NC virtual structure sheaves. The quasi NCDG structure in Theorem 1.2 is interpreted as a smooth dg-resolution of the quasi NC structure (1.2.4). For simplicity, let us assume that the quasi NC structure $\mathcal{O}_{U_i, \bullet}^{\text{nc}}$ in Theorem 1.2 glue to give a global sheaf of non-commutative dg-algebras $\mathcal{O}_{N_\alpha, \bullet}^{\text{nc}}$ on N_α , i.e. the isomorphisms $\phi_{ij, \bullet}$ satisfy the cocycle condition. As an analogy of the identity (1.1.4), one may

try to define the NC virtual structure sheaf by

$$(1.4.1) \quad \mathcal{O}_{M_\alpha}^{\text{ncvir}} = \sum_{i \in \mathbb{Z}} (-1)^i [\mathcal{H}_i(\mathcal{O}_{N_\alpha, \bullet}^{\text{nc}})].$$

The issue of the above construction is that the sum (1.4.1) may be an infinite sum, so does not make sense, even if the condition (1.1.2) is satisfied.

Instead of (1.4.1), if the condition (1.1.2) is satisfied, the following sum turns out to be finite for each $d \in \mathbb{Z}_{\geq 0}$:

$$(1.4.2) \quad (\mathcal{O}_{M_\alpha}^{\text{ncvir}})^{\leq d} := \sum_{i \in \mathbb{Z}} (-1)^i [\mathcal{H}_i((\mathcal{O}_{N_\alpha, \bullet}^{\text{nc}})^{\leq d})].$$

Here $(\mathcal{O}_{N_\alpha, \bullet}^{\text{nc}})^{\leq d}$ is the quotient of $\mathcal{O}_{N_\alpha, \bullet}^{\text{nc}}$ by its d -th step NC filtration (cf. Subsection 2.3). The quotient $(\mathcal{O}_{N_\alpha, \bullet}^{\text{nc}})^{\leq d}$ is interpreted as a d -smooth dg-resolution of the quasi NC structure $\{(V_i, (\mathcal{O}_{V_i}^{\text{nc}})^{\leq d})\}_{i \in \mathbb{I}}$ on M_α , hence (1.4.2) is regarded as a d -smooth thickening of (1.1.3). Moreover the sum (1.4.2) also makes sense even if the quasi NCDG structure in Theorem 1.2 does not satisfy the cocycle condition (cf. Definition 2.14). We call the sum (1.4.2) as d -th NC virtual structure sheaf of the quasi NC structure (1.2.4).

1.5. Descriptions via perfect obstruction theories. We will prove that the d -th NC virtual structure sheaf (1.4.2) is described in terms of the usual virtual structure sheaf (1.1.3) together with the perfect obstruction theory $\mathcal{E}_\bullet \rightarrow \tau_{\geq -1} \mathbf{L}_{M_\alpha}$ induced by the cotangent complex of the commutative dg-scheme $(N_\alpha, \mathcal{O}_{N_\alpha, \bullet})$. We have the following result:

Theorem 1.3. (Theorem 3.9) *We have the following formula*

$$(1.5.1) \quad (\mathcal{O}_{M_\alpha}^{\text{ncvir}})^{\leq d} = \mathcal{O}_{M_\alpha}^{\text{vir}} \otimes_{\mathcal{O}_{M_\alpha}} [S_{\mathcal{O}_{M_\alpha}} L_{\mathcal{O}_{M_\alpha}}^+(\mathcal{E}_\bullet)^{\leq d}].$$

Here $L_{\mathcal{O}_M}(\mathcal{E}_\bullet)$ is the sheaf of super Lie algebras in $T_{\mathcal{O}_M}(\mathcal{E}_\bullet)$ generated by \mathcal{E}_\bullet , and $S_{\mathcal{O}_M}(-)$ is the super symmetric product over \mathcal{O}_M . We refer to Subsection 3.4 for details of the notation of the RHS of (1.5.1). The formula (1.5.1) implies that (1.4.2) is described using the perfect obstruction theory, without referring to quasi NCDG structures. Also it is described by Schur complexes $\mathbf{S}_\lambda(\mathcal{E}_\bullet)$ for partitions λ . For example in the $d = 2$ case, the RHS of (1.5.1) is written as (cf. Corollary 3.11)

$$\mathcal{O}_{M_\alpha}^{\text{vir}} \otimes_{\mathcal{O}_{M_\alpha}} (1 + \mathbf{S}_{(1,1)}(\mathcal{E}_\bullet) + \mathbf{S}_{(2,1)}(\mathcal{E}_\bullet) + \mathbf{S}_{(2,2)}(\mathcal{E}_\bullet) + \mathbf{S}_{(1,1,1,1)}(\mathcal{E}_\bullet)).$$

In particular, the formula (1.5.1) implies that $(\mathcal{O}_{M_\alpha}^{\text{ncvir}})^{\leq d} = \mathcal{O}_{M_\alpha}^{\text{vir}}$ if the expected dimension of M_α is zero. This implies that, if X is a Calabi-Yau 3-fold, the integrations of NC virtual structure sheaves coincide with the usual (commutative) DT invariants. On the other hand, if we consider moduli spaces of stable sheaves on algebraic surfaces or Fano 3-folds so that they have the positive expected dimensions, the resulting NC virtual structure sheaves are in general different from the commutative virtual structure sheaves. In such cases, integrations of their Chern characters may yield interesting enumerative invariants.

In the next paper [Toda], we will pursue another approach in constructing interesting enumerative invariants of sheaves involving non-commutative structures on the moduli spaces of stable sheaves. We can consider motives of Hilbert schemes of points on a quasi NC structure (1.2.4), and construct certain enumerative invariants by integrating the Behrend functions on them. If X is a Calabi-Yau 3-fold, using wall-crossing argument, we can relate these invariants with generalized DT

invariants counting semistable sheaves on X [JS12], [KS] whose definition involves motivic Hall algebras. This would give an intrinsic understanding of the dimension formula [Todc] of Donovan-Wemyss's non-commutative widths [DW] for floppable rational curves, whose detail will be included in [HT].

1.6. Plan of the paper. In Section 2, we introduce the notion of quasi NCDG structures on commutative dg-schemes, and use it to define the NC virtual structure sheaves. In Section 3, we describe the NC virtual structure sheaves via perfect obstruction theory, and prove Theorem 1.5.1. In Section 4, we construct quasi NCDG structures on the moduli spaces of representations of a certain quiver. In Section 5, using the result of Section 4, we prove Theorem 1.2.

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1.8. Notation and convention. In this paper, an algebra always means an associative, not necessary commutative, \mathbb{C} -algebra. The tensor product \otimes is over \mathbb{C} if there is no subscript. Also all the varieties or schemes are defined over \mathbb{C} .

2. NC VIRTUAL STRUCTURE SHEAVES

In this section, we recall virtual structure sheaves associated to commutative dg-schemes, and introduce its NC version. We prepare the following convention on the super symmetric product. For a graded vector space W_\bullet , let S_n acts on $W_\bullet^{\otimes n}$ in the super sense, i.e. the action of the permutation $(i, j) \in S_n$ is

$$\begin{aligned} x_1 \otimes \cdots \otimes x_i \otimes \cdots \otimes x_j \otimes \cdots \otimes x_n \\ \mapsto (-1)^{\deg x_i \deg x_j} x_1 \otimes \cdots \otimes x_j \otimes \cdots \otimes x_i \otimes \cdots \otimes x_n \end{aligned}$$

for homogeneous elements $x_1, \dots, x_n \in W_\bullet$. The *super symmetric product* of W_\bullet is defined by

$$S(W_\bullet) := \bigoplus_{n \geq 0} (W_\bullet^{\otimes n})^{S_n}.$$

The above super symmetric product is obviously generalized for graded vector bundles \mathcal{Q}_\bullet on a scheme N , and we obtain the sheaf of super commutative graded algebras $S_{\mathcal{O}_N}(\mathcal{Q}_\bullet)$ on N . Here a graded algebra A_\bullet is called *super graded commutative* if $a_e \cdot a_{e'} = (-1)^{ee'} a_{e'} \cdot a_e$ for $a_e \in A_e, a_{e'} \in A_{e'}$.

2.1. Commutative dg-schemes. Let $(N, \mathcal{O}_{N, \bullet})$ be a smooth commutative dg-scheme, i.e. N is a smooth scheme and $\mathcal{O}_{N, \bullet}$ is a sheaf of super commutative dg-algebras of the form¹

$$\mathcal{O}_{N, \bullet} = S_{\mathcal{O}_N}(\mathcal{Q}_{-1} \oplus \cdots \oplus \mathcal{Q}_{-k})$$

¹It may be more natural to use the upper index \mathcal{O}_N^\bullet to denote the grading of the sheaf of dg-algebras. In this paper, we use the lower index $\mathcal{O}_{N, \bullet}$ as we will use several other upper gradings.

for vector bundles \mathcal{Q}_i on N located in degree i . The zero-th cohomology of $\mathcal{O}_{N,\bullet}$ is written as \mathcal{O}_N/J for the ideal sheaf $J \subset \mathcal{O}_N$, hence determines a closed subscheme $M \subset N$. We write

$$\tau_0(N, \mathcal{O}_{N,\bullet}) := M$$

and call it the *zero-th truncation* of $(N, \mathcal{O}_{N,\bullet})$.

Let E_\bullet be a finitely generated dg- $\mathcal{O}_{N,\bullet}$ -module. We will use two kinds of restrictions of it to M

$$(2.1.1) \quad E_\bullet|_M := E_\bullet \otimes_{\mathcal{O}_N} \mathcal{O}_M, \quad \overline{E}_\bullet|_M := E_\bullet \otimes_{\mathcal{O}_{N,\bullet}} \mathcal{O}_M.$$

By setting $\mathbf{O}_\bullet := \mathcal{O}_{N,\bullet}|_M$, the two restrictions (2.1.1) are related by

$$\overline{E}_\bullet|_M = E_\bullet|_M / (\mathbf{O}_{\leq -1} E_\bullet|_M).$$

Let $\Omega_{N,\bullet}$ be the cotangent complex of $(N, \mathcal{O}_{N,\bullet})$. The complex $\overline{\Omega}_{N,\bullet}|_M$ is described as

$$\overline{\Omega}_{N,\bullet}|_M = \left(0 \rightarrow \mathcal{Q}_{-k}|_M \rightarrow \cdots \rightarrow \mathcal{Q}_{-2}|_M \xrightarrow{d_{-1}} \mathcal{Q}_{-1}|_M \xrightarrow{d_0} \Omega_N|_M \rightarrow 0 \right).$$

Let $T_{N,\bullet} := \mathcal{H}om_{\mathcal{O}_{N,\bullet}}(\Omega_{N,\bullet}, \mathcal{O}_{N,\bullet})$ be the tangent complex of $(N, \mathcal{O}_{N,\bullet})$.

Definition 2.1. A smooth commutative dg-scheme $(N, \mathcal{O}_{N,\bullet})$ is called a $[0, 1]$ -manifold if the cohomologies of the complex $\overline{T}_{N,\bullet}|_M$ are concentrated on $[0, 1]$.

Suppose that $(N, \mathcal{O}_{N,\bullet})$ is a $[0, 1]$ -manifold. Then $\overline{T}_{N,\bullet}|_M$ is quasi-isomorphic to the complex

$$(2.1.2) \quad 0 \rightarrow T_N|_M \xrightarrow{(d_0)^\vee} K \rightarrow 0$$

where K is the kernel of $(d_{-1}|_M)^\vee: (\mathcal{Q}_{-1}|_M)^\vee \rightarrow (\mathcal{Q}_{-2}|_M)^\vee$, which is a locally free sheaf on N . We define \mathcal{E}_\bullet to be the dual of the complex (2.1.2)

$$(2.1.3) \quad \mathcal{E}_\bullet := (0 \rightarrow K^\vee \xrightarrow{\overline{d}_0} \Omega_N|_M \rightarrow 0)$$

where $\Omega_N|_M$ is located in degree zero and \overline{d}_0 is induced by d_0 . We have the quasi-isomorphism $\overline{\Omega}_{N,\bullet}|_M \xrightarrow{\sim} \mathcal{E}_\bullet$, and by [CFK09, Proposition 3.2.4], we have the morphism of complexes

$$(2.1.4) \quad \mathcal{E}_\bullet \rightarrow (0 \rightarrow J/J^2 \rightarrow \Omega_N|_M \rightarrow 0)$$

giving a perfect obstruction theory on M in the sense of Behrend-Fantechi [BF97].

2.2. Commutative virtual structure sheaves. For a $[0, 1]$ -manifold $(N, \mathcal{O}_{N,\bullet})$, let us recall the virtual fundamental class and its K-theoretic enhancement associated to the perfect obstruction theory (2.1.4). Let $C_{M/N}$ be the normal cone of M in N defined by

$$C_{M/N} := \operatorname{Spec}_{\mathcal{O}_M} \bigoplus_{k \geq 0} J^k / J^{k+1}.$$

By [CFK09, Proposition 1.2.1], we have the closed embedding $C_{M/N} \subset K$, hence obtain the following diagram

$$\begin{array}{ccc} C_{M/N} & \hookrightarrow & K \\ & \searrow & \nearrow j \\ & M. & \end{array}$$

Here j is the zero section. The virtual fundamental class associated to (2.1.4) is defined by

$$(2.2.1) \quad [M]^{\text{vir}} := j^! [C_{M/N}] \in A_{\bullet}(M).$$

By the above definition, the virtual fundamental class has a K-theoretic enhancement, called the *virtual structure sheaf*

$$(2.2.2) \quad \mathcal{O}_M^{\text{vir}} := [\mathbf{L}j^* \mathcal{O}_{C_{M/N}}] \in K_0(M).$$

Note that the virtual structure sheaf recovers the virtual fundamental class by

$$\text{cl}(\mathcal{O}_M^{\text{vir}}) = [M]^{\text{vir}} \in A_{\bullet}(M).$$

Here cl is the cycle map.

On the other hand, note that each cohomology sheaf $\mathcal{H}_i(\mathcal{O}_{N,\bullet})$ is a coherent \mathcal{O}_M -module, which vanishes for $i \ll 0$ by [CFK09, Theorem 2.2.2]. Hence for a finitely generated $\text{dg-}\mathcal{O}_{N,\bullet}$ -module E_{\bullet} , each cohomology $\mathcal{H}_i(E_{\bullet})$ is a coherent \mathcal{O}_M -module, and vanishes for $|i| \gg 0$. Therefore the following definition makes sense:

$$[E_{\bullet}] := \sum_{i \in \mathbb{Z}} (-1)^i [\mathcal{H}_i(E_{\bullet})] \in K_0(M).$$

The above K-theory classes from the dg-schemes are related to the virtual structure sheaf as follows:

Theorem 2.2. ([CFK09, Theorem 4.4.2]) *Let E_{\bullet} be a finitely generated locally free $\text{dg-}\mathcal{O}_{N,\bullet}$ -module. Then we have the equality in $K_0(M)$:*

$$[E_{\bullet}] = \mathcal{O}_M^{\text{vir}} \otimes_{\mathcal{O}_M} [\overline{E}_{\bullet}|_M].$$

In particular, we have the identity $\mathcal{O}_M^{\text{vir}} = [\mathcal{O}_{N,\bullet}]$ in $K_0(M)$.

Note that for a (not necessary differential) graded $\mathcal{O}_{N,\bullet}$ -module F_{\bullet} , we can similarly define the restriction $\overline{F}_{\bullet}|_M = F_{\bullet} \otimes_{\mathcal{O}_{N,\bullet}} \mathcal{O}_M$. Let

$$(2.2.3) \quad [F_{\bullet}] \in K_0(\mathcal{O}_{N,\bullet})$$

be its class in the K-group of finitely generated graded $\mathcal{O}_{N,\bullet}$ -modules. We have the following corollary of Theorem 2.2:

Corollary 2.3. *In the situation of Theorem 2.2, let F_{\bullet} be a locally free graded $\mathcal{O}_{N,\bullet}$ -module. Suppose that $[F_{\bullet}] = [E_{\bullet}]$ in $K_0(\mathcal{O}_{N,\bullet})$. Then we have the identity*

$$[E_{\bullet}] = \mathcal{O}_M^{\text{vir}} \otimes_{\mathcal{O}_M} [\overline{F}_{\bullet}|_M].$$

Proof. The corollary follows from Theorem 2.2 since $[\overline{E}_{\bullet}|_M] \in K_0(M)$ is independent of the differential on E_{\bullet} . \square

2.3. Graded NC filtrations. We introduce the non-commutative version of some notions recalled in the previous subsections. Let R be an algebra which is not necessary commutative, and W_{\bullet} a finite dimensional graded vector space with $W_i = 0$ for $i \geq 0$. Below, we call a grading induced by the grading on W_{\bullet} as $|\bullet$ -grading. We set the $|\bullet$ -graded algebra Λ_{\bullet} to be

$$(2.3.1) \quad \Lambda_{\bullet} := R * T(W_{\bullet}).$$

Here $T(W_{\bullet})$ is the tensor algebra

$$T(W_{\bullet}) := \bigoplus_{n \geq 0} W_{\bullet}^{\otimes n}$$

and $*$ is the free product as \mathbb{C} -algebras. Note that

$$\Lambda_{>0} = 0, \Lambda_0 = \Lambda, \Lambda_{-1} = R \otimes W_{-1} \otimes R$$

and so on. We regard Λ_\bullet as a $|\bullet$ -graded super Lie algebra by setting

$$[x, y] := xy - (-1)^{ab}yx, \quad x \in \Lambda_a, \quad y \in \Lambda_b.$$

The subspace $\Lambda_{\bullet, k}^{\text{Lie}} \subset \Lambda_\bullet$ is defined to be spanned by the elements of the form

$$[x_1, [x_2, \dots, [x_{k-1}, x_k] \dots]]$$

for $x_i \in \Lambda_\bullet$, $1 \leq i \leq k$. The $|\bullet$ -graded NC filtration of Λ_\bullet is the decreasing filtration

$$(2.3.2) \quad \Lambda_\bullet = F^0 \Lambda_\bullet \supset F^1 \Lambda_\bullet \supset \dots \supset F^d \Lambda_\bullet \supset \dots$$

where $F^d \Lambda_\bullet$ is the two-sided $|\bullet$ -graded ideal of Λ_\bullet defined by

$$F^d \Lambda_\bullet := \sum_{m \geq 0} \sum_{i_1 + \dots + i_m = m+d} \Lambda_\bullet \cdot \Lambda_{\bullet, i_1}^{\text{Lie}} \cdot \Lambda_\bullet \cdot \dots \cdot \Lambda_{\bullet, i_m}^{\text{Lie}} \cdot \Lambda_\bullet.$$

Note that $\Lambda_\bullet / F^1 \Lambda_\bullet$ is the abelization Λ_\bullet^{ab} of Λ_\bullet , which is a super commutative $|\bullet$ -graded algebra written as

$$(2.3.3) \quad \Lambda_\bullet^{ab} = R^{ab} \otimes S(W_\bullet).$$

We set $\Lambda_\bullet^{\leq d} := \Lambda_\bullet / F^{d+1} \Lambda_\bullet$, and $N_\bullet^{\leq d} := \Lambda_\bullet^{\leq d} \otimes_{\Lambda_\bullet} N_\bullet$ for a graded left Λ_\bullet -module N_\bullet . By the definition of the filtration (2.3.2), the subquotient

$$(2.3.4) \quad \text{gr}_F(\Lambda_\bullet) := \bigoplus_{d \geq 0} F^d \Lambda_\bullet / F^{d+1} \Lambda_\bullet$$

is a bi-graded algebra: it is a direct sum of

$$\text{gr}_F(\Lambda_\bullet)_e^d := (F^d \Lambda_\bullet / F^{d+1} \Lambda_\bullet)_e$$

where e is the $|\bullet$ -grading, and we call the degree d as $|\bullet$ -grading.

2.4. Quasi NC structures. In the notation of the previous subsection, suppose that $W_\bullet = 0$ so that $\Lambda_\bullet = R$ holds. We recall some notions on NC algebras following [Kap98].

Definition 2.4. (i) An algebra R is called NC nilpotent of degree d (resp. NC nilpotent) if $F^{d+1} R = 0$ (resp. $F^n R = 0$ for $n \gg 0$).

(ii) An algebra R is called NC complete if the following natural map is an isomorphism

$$R \rightarrow R_{[ab]} := \varprojlim R^{\leq d}.$$

(iii) An NC nilpotent algebra R is called of finite type if R^{ab} is a finitely generated \mathbb{C} -algebra and each $\text{gr}_F^d(R)$ is a finitely generated R^{ab} -module.

Let R be an NC complete algebra. For any multiplicative set $S \subset R^{ab}$ without zero divisor, its pull-back by the natural surjection $R^{\leq d} \twoheadrightarrow R^{ab}$ determines the multiplicative set in $R^{\leq d}$, which satisfies the Ore localization condition (cf. [Kap98, Proposition 2.1.5]). In particular, one can define the localization $S^{-1} R^{\leq d}$ of $R^{\leq d}$ by S . Therefore, similarly to the case of usual affine schemes, the NC nilpotent algebra $R^{\leq d}$ determines the sheaf of algebras $\tilde{R}^{\leq d}$ on $\text{Spec } R^{ab}$ (cf. [Kap98, Definition 2.2.2]). Namely, the topological basis of $\text{Spec } R^{ab}$ is given by

$$U_f := \{\mathfrak{p} \in \text{Spec } R^{ab} : f \notin \mathfrak{p}\}$$

and $\tilde{R}^{\leq d}$ is the sheafification of the presheaf $U_f \mapsto (f)^{-1}R^{\leq d}$, where (f) is the multiplicative set $\{f^n : n \geq 0\}$ in R^{ab} . Similarly, for any left $R^{\leq d}$ -module P , the sheaf \tilde{P} is defined to be the sheafification of the presheaf

$$U_f \mapsto (f)^{-1}R^{\leq d} \otimes_{R^{\leq d}} P.$$

The ringed space

$$\mathrm{Spf} R := (\mathrm{Spec} R^{ab}, \tilde{R}), \quad \tilde{R} := \varprojlim \tilde{R}^{\leq d}$$

is called an *affine NC scheme*, or an *affine NC structure* on the affine scheme $\mathrm{Spec} R^{ab}$. The sheaf \tilde{R} is determined by the localization for a multiplicative set $S \subset R^{ab}$, given by

$$S^{-1}R := \varprojlim S^{-1}R^{\leq d}.$$

Definition 2.5. ([Kap98]) *A ringed space is called an NC scheme if it is locally isomorphic to affine NC schemes.*

For a scheme M , an *NC structure* is an NC scheme $(M, \mathcal{O}_M^{\mathrm{nc}})$ with $(\mathcal{O}_M^{\mathrm{nc}})^{ab} = \mathcal{O}_M$. In [Toddb], a weaker notion of the NC structures was considered:

Definition 2.6. ([Toddb]) *Let M be a commutative scheme. A quasi NC structure on M consists of an affine open cover $\{V_i\}_{i \in \mathbb{I}}$ of M , affine NC structures $V_i^{\mathrm{nc}} = (V_i, \mathcal{O}_{V_i}^{\mathrm{nc}})$ on V_i for each $i \in \mathbb{I}$, and isomorphisms of NC schemes*

$$\phi_{ij} : (V_{ij}, \mathcal{O}_{V_{ij}}^{\mathrm{nc}}|_{V_{ij}}) \xrightarrow{\cong} (V_{ij}, \mathcal{O}_{V_i}^{\mathrm{nc}}|_{V_{ij}})$$

satisfying $\phi_{ij}^{ab} = id$.

Let \mathcal{N}_d be the category of NC nilpotent algebras of degree d , and \mathcal{N} the category of NC nilpotent algebras. An exact sequence

$$(2.4.1) \quad 0 \rightarrow J \rightarrow R_1 \rightarrow R_2 \rightarrow 0$$

in \mathcal{N} is called a *central extension* if $J^2 = 0$ and J lies in the center of R_1 .

Definition 2.7. (i) *An NC nilpotent algebra R of degree d is called d -smooth if it is of finite type and the functor*

$$h_R := \mathrm{Hom}(R, -) : \mathcal{N} \rightarrow \mathrm{Set}$$

is formally d -smooth, i.e. for any central extension (2.4.1) in \mathcal{N}_d , the map $h_R(R_1) \rightarrow h_R(R_2)$ is surjective.

(ii) *An NC complete algebra R is called smooth if $R^{\leq d}$ is d -smooth for any $d \geq 0$.*

A quasi NC structure in Definition 2.6 is called smooth if each U_i^{nc} is written as $U_i^{\mathrm{nc}} = \mathrm{Spf} R_i$ for a smooth algebra R_i . If M admits a smooth quasi NC structure, then M must be smooth. Conversely, any smooth variety admits a smooth quasi NC structure by [Kap98, Theorem 1.6.1].

2.5. Quasi NCDG structures. Let R be an NC complete algebra and Λ_\bullet a graded algebra given by (2.3.1). Suppose that there is a degree one differential

$$Q : \Lambda_\bullet \rightarrow \Lambda_{\bullet+1}$$

giving a dg-algebra structure on Λ_\bullet . By the Leibniz rule, the differential Q preserves the filtration (2.3.2), hence we have the induced dg-algebra $(\Lambda_\bullet^{\leq d}, Q^{\leq d})$. Note that

each $|\bullet$ -degree term of $\Lambda_\bullet^{\leq d}$ is a left $R^{\leq d}$ -module, and $Q^{\leq d}$ is a left $R^{\leq d}$ -module homomorphism. Hence we have the associated sheaf of dg-algebras $\tilde{\Lambda}_\bullet^{\leq d}$ on $\text{Spec } R^{ab}$, which is a complex of quasi-coherent left $\tilde{R}^{\leq d}$ -modules, and the dg-ringed space

$$\text{Spf } \Lambda_\bullet^{\leq d} := (\text{Spec } R^{ab}, \tilde{\Lambda}_\bullet^{\leq d}).$$

We see that the above dg-ringed space is also locally written of the above form.

Lemma 2.8. *For any multiplicative set $S \subset R^{ab}$ without zero divisor, we have the canonical isomorphism*

$$(2.5.1) \quad S^{-1}R^{\leq d} \otimes_{R^{\leq d}} \Lambda_\bullet^{\leq d} \xrightarrow{\cong} (S^{-1}R * T(W_\bullet))^{\leq d}.$$

Proof. Note that there exists a canonical morphism from the LHS to the RHS of (2.5.1) by the universality of the localization. We prove the isomorphism (2.5.1) by the induction of d . For $d = 0$, the claim is obvious since both sides coincide with $S^{-1}R^{ab} \otimes S(W_\bullet)$. Suppose that the isomorphism (2.5.1) holds for $d \geq 0$. Since $S^{-1}R^{\leq d+1}$ is a flat right $R^{\leq d+1}$ -module, we have the exact sequence

$$(2.5.2) \quad \begin{aligned} 0 \rightarrow S^{-1}R^{ab} \otimes_{R^{ab}} \text{gr}_F(\Lambda_\bullet)^d \rightarrow S^{-1}R^{\leq d+1} \otimes_{R^{\leq d+1}} \Lambda_\bullet^{\leq d+1} \\ \rightarrow S^{-1}R^{\leq d} \otimes_{R^{\leq d}} \Lambda_\bullet^{\leq d} \rightarrow 0. \end{aligned}$$

On the other hand, we have the exact sequence

$$(2.5.3) \quad \begin{aligned} 0 \rightarrow \text{gr}_F(S^{-1}R * T(W_\bullet))^d \rightarrow (S^{-1}R * T(W_\bullet))^{\leq d+1} \\ \rightarrow (S^{-1}R * T(W_\bullet))^{\leq d} \rightarrow 0. \end{aligned}$$

By the assumption of the induction, (2.5.2), (2.5.3), and the five lemma, it is enough to show the isomorphism

$$S^{-1}R^{ab} \otimes_{R^{ab}} \text{gr}_F(\Lambda_\bullet)^d \xrightarrow{\cong} \text{gr}_F(S^{-1}R * T(W_\bullet))^d.$$

In Subsection 3.3, we will see that the subquotients of the NC filtrations are described by Poisson envelopes. Using Lemma 3.8 in Subsection 3.3, it is enough to show the isomorphism

$$S^{-1}R^{ab} \otimes_{R^{ab}} P(R^{ab} \otimes S(W_\bullet))^d \xrightarrow{\cong} P(S^{-1}R^{ab} \otimes S(W_\bullet))^d.$$

The above isomorphism follows since taking the Poisson envelope commutes with the localization (cf. [Kap98, Section 4.1]). \square

We define the following dg-ringed space

$$(2.5.4) \quad \text{Spf } \Lambda_\bullet := (\text{Spec } R^{ab}, \tilde{\Lambda}_\bullet), \quad \tilde{\Lambda}_\bullet := \varprojlim \tilde{\Lambda}_\bullet^{\leq d}.$$

The sheaf $\tilde{\Lambda}_\bullet$ is a sheaf of dg-algebras on $\text{Spec } R^{ab}$, which is a complex of left \tilde{R} -modules. Note that

$$(2.5.5) \quad \text{Spec } \Lambda_\bullet^{ab} := (\text{Spec } R^{ab}, (\tilde{\Lambda}_\bullet)^{ab})$$

is an affine commutative dg-scheme. We call the dg-ringed space (2.5.4) as an *affine NCDG scheme* or an *affine NCDG structure* on the commutative dg-scheme (2.5.5). We call it *smooth* if the ungraded algebra R is smooth. In this case, (2.5.5) is a smooth affine commutative dg-scheme. The *zero-th truncation* of (2.5.4) is defined by

$$\tau_0(\text{Spf } \Lambda_\bullet) := (\text{Spec } \mathcal{H}_0(\Lambda_\bullet^{ab}), \varprojlim \mathcal{H}_0(\tilde{\Lambda}_\bullet^{\leq d})).$$

Since $\mathcal{H}_0(\Lambda_\bullet^{\leq d}) = \mathcal{H}_0(\Lambda_\bullet)^{\leq d}$, we have $\tau_0(\mathrm{Spf} \Lambda_\bullet) = \mathrm{Spf} \mathcal{H}_0(\Lambda_\bullet)$, which is an affine NC structure on $\mathrm{Spec} \mathcal{H}_0(\Lambda_\bullet^{ab})$.

Example 2.9. Let $R = \mathbb{C}[x]$ and set $W_\bullet = W_{-1} = \mathbb{C} \cdot y$. Then $\Lambda_\bullet = \mathbb{C}\langle x, y \rangle$ where x is degree zero and y is degree -1 . For $n \geq 1$, let Q be the differential given by

$$(2.5.6) \quad Q: \Lambda_\bullet \rightarrow \Lambda_{\bullet+1}, \quad Q(x) = 0, \quad Q(y) = x^n.$$

We have the associated affine NCDG scheme

$$(2.5.7) \quad \mathrm{Spf} \Lambda_\bullet = (\mathrm{Spec} \mathbb{C}[x], \widetilde{\mathbb{C}\langle x, y \rangle})$$

which is an affine NCDG structure on the commutative dg-scheme

$$(2.5.8) \quad \mathrm{Spec} \Lambda_\bullet^{ab} = (\mathrm{Spec} \mathbb{C}[x], \widetilde{\mathbb{C}[x]} \oplus \widetilde{\mathbb{C}[x]y}).$$

The zero-th truncation of (2.5.7) is $\mathrm{Spec} \mathbb{C}[x]/(x^n)$.

We define the following NCDG analogue of NC schemes.

Definition 2.10. A dg-ringed space $(N, \mathcal{O}_{N, \bullet}^{\mathrm{nc}})$ is called an NCDG scheme if it is locally isomorphic to affine NCDG schemes.

By Lemma 2.8, for any open subset $U \subset \mathrm{Spec} R^{ab}$, the restriction $(U, \widetilde{\Lambda}_\bullet|_U)$ is an NCDG scheme. We can also define the NCDG analogue of Definition 2.6:

Definition 2.11. Let $(N, \mathcal{O}_{N, \bullet})$ be a commutative dg-scheme. A quasi NCDG structure on $(N, \mathcal{O}_{N, \bullet})$ consists of an affine open cover $\{U_i\}_{i \in \mathbb{I}}$ of N , affine NCDG structures $(U_i, \mathcal{O}_{U_i, \bullet}^{\mathrm{nc}})$ on $(U_i, \mathcal{O}_{N, \bullet}|_{U_i})$ for each $i \in \mathbb{I}$, and isomorphisms of NCDG schemes

$$(2.5.9) \quad \phi_{ij, \bullet}: (U_{ij}, \mathcal{O}_{U_{ij}, \bullet}^{\mathrm{nc}}|_{U_{ij}}) \xrightarrow{\cong} (U_{ij}, \mathcal{O}_{U_i, \bullet}^{\mathrm{nc}}|_{U_{ij}}).$$

satisfying $(\phi_{ij, \bullet})^{ab} = id$.

A quasi NCDG structure in Definition 2.11 is called *smooth* if each $(U_i, \mathcal{O}_{U_i, \bullet}^{\mathrm{nc}})$ is smooth. If $(N, \mathcal{O}_{N, \bullet})$ admits a smooth quasi NCDG structure, then it must be smooth. If the isomorphisms (2.5.9) satisfy the cocycle condition, the sheaves of dg-algebras $\mathcal{O}_{U_i, \bullet}^{\mathrm{nc}}$ glue to give the sheaf of dg-algebras $\mathcal{O}_{N, \bullet}^{\mathrm{nc}}$ on N . In this case, a pair $(N, \mathcal{O}_{N, \bullet}^{\mathrm{nc}})$ is an NCDG scheme, and called a *NCDG structure* on $(N, \mathcal{O}_{N, \bullet})$. Let $M \subset N$ be the closed subscheme given by the zero-th truncation of $(N, \mathcal{O}_{N, \bullet})$, and set $V_i = M \cap U_i$ for the quasi NCDG structure in Definition 2.11. By the definition, we have the induced quasi NC structure

$$(2.5.10) \quad \{V_i^{\mathrm{nc}} = \tau_0(U_i, \mathcal{O}_{U_i, \bullet}^{\mathrm{nc}})\}_{i \in \mathbb{I}}, \quad \mathcal{H}_0(\phi_{ij, \bullet}): V_j^{\mathrm{nc}}|_{V_{ij}} \xrightarrow{\cong} V_i^{\mathrm{nc}}|_{V_{ij}}$$

on M in the sense of Definition 2.6. We call the quasi NC structure (2.5.10) as the *zero-th truncation* of the quasi NCDG structure in Definition 2.11.

2.6. NC virtual structure sheaves. Let $(N, \mathcal{O}_{N, \bullet})$ be a smooth commutative dg-scheme, which admits a smooth quasi NCDG structure in Definition 2.11. For simplicity, suppose that the quasi NCDG structure glues to give an NCDG structure $(N, \mathcal{O}_{N, \bullet}^{\mathrm{nc}})$ on $(N, \mathcal{O}_{N, \bullet})$. Then its zero-th truncation is an NC structure $M^{\mathrm{nc}} = (M, \mathcal{O}_M^{\mathrm{nc}})$ on M . As an analogy of the identity $\mathcal{O}_M^{\mathrm{vir}} = \llbracket \mathcal{O}_{N, \bullet} \rrbracket$ in Theorem 2.2, one would like to define the ‘NC virtual structure sheaf’ of M^{nc} to be

$$(2.6.1) \quad \mathcal{O}_M^{\mathrm{ncvir}} = \sum_{i \in \mathbb{Z}} (-1)^i [\mathcal{H}_i(\mathcal{O}_{N, \bullet}^{\mathrm{nc}})].$$

Note that each $\mathcal{H}_i(\mathcal{O}_{N,\bullet}^{\text{nc}})$ is a quasi coherent left $\mathcal{O}_M^{\text{nc}}$ -module. Hence if they are coherent and vanish for $|i| \gg 0$, then the sum (2.6.1) makes sense as an element of $K_0(M^{\text{nc}})$, where $K_0(M^{\text{nc}})$ is the Grothendieck group of the category of coherent left $\mathcal{O}_M^{\text{nc}}$ -modules. However in general, the sum (2.6.1) is an infinite sum even if $(N, \mathcal{O}_{N,\bullet})$ is a $[0, 1]$ -manifold. For example, the cohomologies of the complex (2.5.6) are not bounded while the commutative dg-scheme (2.5.8) is a $[0, 1]$ -manifold.

Instead of $(N, \mathcal{O}_{N,\bullet}^{\text{nc}})$, consider the NCDG scheme $(N, (\mathcal{O}_{N,\bullet}^{\text{nc}})^{\leq d})$ for $d \in \mathbb{Z}_{\geq 0}$. It is regarded as a d -smooth dg-resolution of the d -th order NC thickening $(M^{\text{nc}})^{\leq d} = (M, (\mathcal{O}_M^{\text{nc}})^{\leq d})$ of M . Moreover we have the following:

Lemma 2.12. *If $(N, \mathcal{O}_{N,\bullet})$ is a $[0, 1]$ -manifold, then $(\mathcal{O}_{N,\bullet}^{\text{nc}})^{\leq d}$ is quasi-isomorphic to a bounded complex.*

Proof. The subquotient of the NC filtration of $(\mathcal{O}_{N,\bullet}^{\text{nc}})^{\leq d}$ is the direct sum of $\text{gr}_F(\mathcal{O}_{N,\bullet}^{\text{nc}})^j$ for $0 \leq j \leq d$. It is easy to see that $\text{gr}_F(\mathcal{O}_{N,\bullet}^{\text{nc}})^j$ is a finitely generated dg $\mathcal{O}_{N,\bullet}$ -module, hence bounded if $(N, \mathcal{O}_{N,\bullet})$ is a $[0, 1]$ -manifold. Therefore the lemma holds. \square

By the above lemma, the sum

$$(2.6.2) \quad (\mathcal{O}_M^{\text{ncvir}})^{\leq d} = \sum_{i \in \mathbb{Z}} (-1)^i [\mathcal{H}_i((\mathcal{O}_{N,\bullet}^{\text{nc}})^{\leq d})]$$

makes sense in $K_0((M^{\text{nc}})^{\leq d})$, which is identified as an element of $K_0(M)$ since any left $(\mathcal{O}_M^{\text{nc}})^{\leq d}$ -module has a finite filtration whose subquotients are \mathcal{O}_M -modules. We call (2.6.2) as d -th NC virtual structure sheaf of the d -th NC thickening $(M^{\text{nc}})^{\leq d}$ of M .

In general, a quasi NCDG structure in Definition 2.11 may not glue to give an NCDG structure. In such a case, the zero-th truncation (2.5.10) only gives a quasi NC structure M^{nc} on M . We generalize the above notion of d -th NC virtual structure sheaves to the quasi NC structure M^{nc} . Even if the isomorphisms (2.5.9) do not satisfy the cocycle condition, we have the following lemma:

Lemma 2.13. *The isomorphisms (2.5.9) induce the isomorphisms*

$$\text{gr}_F(\phi_{ij,\bullet})^d_{\bullet} : \text{gr}_F(\mathcal{O}_{U_j,\bullet}^{\text{nc}}|_{U_{ij}})^d_{\bullet} \xrightarrow{\cong} \text{gr}_F(\mathcal{O}_{U_j,\bullet}^{\text{nc}}|_{U_{ij}})^d_{\bullet}$$

of dg- $\mathcal{O}_{N,\bullet}|_{U_{ij}}$ -modules satisfying the cocycle condition.

Proof. Let Λ_{\bullet} be a graded algebra and $\phi_{\bullet} : \Lambda_{\bullet} \rightarrow \Lambda_{\bullet}$ an isomorphism of graded algebras satisfying $\phi_{\bullet}^{ab} = \text{id}$. Then the induced isomorphism $\text{gr}_F(\phi_{\bullet}) : \text{gr}_F(\Lambda_{\bullet}) \rightarrow \text{gr}_F(\Lambda_{\bullet})$ is the identity by [Todb, Lemma 2.2]. The lemma obviously follows from this fact. \square

By the above lemma, the sheaves $\text{gr}_F(\mathcal{O}_{U_i,\bullet}^{\text{nc}})^d_{\bullet}$ glue to give the global dg- $\mathcal{O}_{N,\bullet}$ -module on N

$$(2.6.3) \quad \text{gr}_F(\mathcal{O}_{N,\bullet}^{\text{nc}})^d_{\bullet} \in \text{dg } \mathcal{O}_{N,\bullet}\text{-mod}.$$

Since (2.6.3) is finitely generated, we have the following element

$$[\text{gr}_F(\mathcal{O}_{N,\bullet}^{\text{nc}})^d_{\bullet}] \in K_0(M).$$

Hence the following definition makes sense:

Definition 2.14. Let $(N, \mathcal{O}_N, \bullet)$ be a $[0, 1]$ -manifold and $M \subset N$ its zero-th truncation. Suppose that it admits a quasi NCDG structure, and let M^{nc} be its zero-th truncation (2.5.10). The d -th NC virtual structure sheaf of M^{nc} is defined to be

$$(2.6.4) \quad (\mathcal{O}_M^{\text{ncvir}})^{\leq d} := \sum_{j=0}^d [\text{gr}_F(\mathcal{O}_{N, \bullet}^{\text{nc}})^j] \in K_0(M).$$

Remark 2.15. If a quasi NCDG structure gives the NCDG structure, then the class (2.6.4) coincides with (2.6.2) by taking the NC filtration of $(\mathcal{O}_{N, \bullet}^{\text{nc}})^{\leq d}$.

Remark 2.16. By Theorem 2.2, for $d = 0$ we have the identity $(\mathcal{O}_M^{\text{ncvir}})^{\leq 0} = \mathcal{O}_M^{\text{vir}}$, where $\mathcal{O}_M^{\text{vir}}$ is the commutative virtual structure sheaf given in (2.2.2).

Example 2.17. In the situation of Example 2.9, let $M = \text{Spec } \mathbb{C}[x]/(x^n)$. By definition, the d -th NC virtual structure sheaf of M is

$$(2.6.5) \quad (\mathcal{O}_M^{\text{ncvir}})^{\leq d} = \sum_{j=0}^d [\text{gr}_F(\mathbb{C}\langle x, y \rangle)^j].$$

By the identification $K_0(M) = \mathbb{Z}$, we have $\mathcal{O}_M^{\text{vir}} = [\mathbb{C}[x, y]] = n$ and

$$\begin{aligned} (\mathcal{O}_M^{\text{ncvir}})^{\leq 1} &= [\mathbb{C}[x, y] \oplus \mathbb{C}[x, y][x, y] \oplus \mathbb{C}[x, y][y, y]] \\ &= n. \end{aligned}$$

In general, one can show that (2.6.5) coincides with n for all $d \geq 1$ (cf. Corollary 3.12).

3. DESCRIPTION OF NC VIRTUAL STRUCTURE SHEAVES

In this section, we give an explicit description of the NC virtual structure sheaves in terms of the perfect obstruction theory (2.1.4), and prove Theorem 1.3.

3.1. Graded Poisson envelope. Let Λ_\bullet be a graded algebra (2.3.1), and take the NC filtration (2.3.2). For $x \in (F^d \Lambda_\bullet)_e$, $x' \in (F^{d'} \Lambda_\bullet)_{e'}$, it is easy to see that

$$x \cdot x' \in (F^{d+d'} \Lambda_\bullet)_{e+e'}, \quad [x, x'] \in (F^{d+d'+1} \Lambda_\bullet)_{e+e'}.$$

Here e, e' are $|\bullet|$ -gradings. Therefore the bracket $[-, -]$ induces the pairing

$$\{-, -\}: \text{gr}_F(\Lambda_\bullet)_e^d \times \text{gr}_F(\Lambda_\bullet)_{e'}^{d'} \rightarrow \text{gr}_F(\Lambda_\bullet)_{e+e'}^{d+d'+1}$$

which is super anti-symmetric with respect to the $|\bullet|$ -grading. In general, we introduce the following definition:

Definition 3.1. (i) A $|\bullet|$ -graded Poisson algebra is a triple

$$(3.1.1) \quad (P_\bullet, \cdot, \{-, -\})$$

where (P_\bullet, \cdot) is a super commutative graded algebra, $\{-, -\}$ is a grade preserving, super anti-symmetric pairing

$$(3.1.2) \quad \{-, -\}: P_\bullet \times P_\bullet \rightarrow P_\bullet$$

satisfying the super Jacobi identity and $\{x, -\}$ is a super derivation for any $x \in P_\bullet$.

(ii) A $|\bullet|$ -graded Poisson algebra is a $|\bullet|$ -graded Poisson algebra $(P_\bullet, \cdot, \{-, -\})$ endowed with another grading (called $|\bullet|$ -grading) P_\bullet^\bullet such that the multiplication \cdot preserves the $|\bullet|$ -degree, and the pairing (3.1.2) sends $P_e^d \times P_{e'}^{d'}$ to $P_{e+e'}^{d+d'+1}$.

A $|\bullet$ -graded Poisson algebra P_\bullet satisfying $P_i = 0$ for $i \neq 0$ is nothing but a *Poisson algebra*. It is easy to see that the algebra (2.3.4) is a $|\bullet$ -graded Poisson algebra. Also a $|\bullet$ -graded Poisson algebra $(P_\bullet^\bullet, \cdot, \{-, -\})$ is interpreted as a $|\bullet$ -graded Poisson algebra by forgetting the $|\bullet$ -degree. We consider the functor

$$(3.1.3) \quad (|\bullet\text{-graded Poisson algebras}) \rightarrow (\text{super commutative graded algebras})$$

defined by forgetting (3.1.2), i.e. it sends the triple (3.1.1) to the super commutative graded algebra (P_\bullet, \cdot) . The functor (3.1.3) extends the forgetting functor from the category of Poisson algebras to the category of commutative algebras. It is well-known that the latter functor has a left adjoint, called *Poisson envelope*. We see that the graded version of the similar construction gives the left adjoint of (3.1.3).

For a graded vector space W_\bullet , let

$$(3.1.4) \quad L(W_\bullet) \subset T(W_\bullet)$$

be the $|\bullet$ -graded super Lie algebra generated by W_\bullet . It is a direct sum of $L_e^d(W_\bullet)$, where e is the $|\bullet$ -grading, and d is the grading (called $|\bullet$ -grading) determined by

$$(3.1.5) \quad L_\bullet^0(W_\bullet) = W_\bullet, \quad L_\bullet^{d+1}(W_\bullet) = [W_\bullet, L_\bullet^d(W_\bullet)].$$

For example, $L_e^1(W_\bullet)$ is spanned by

$$[x_0, x_1] = x_0 \otimes x_1 - (-1)^{e_0 e_1} x_1 \otimes x_0, \quad x_i \in W_{e_i}, \quad e_0 + e_1 = e.$$

Definition 3.2. *The free Poisson algebra generated by W_\bullet is defined by*

$$(3.1.6) \quad \text{Poiss}(W_\bullet) := SL(W_\bullet)$$

where $S(-)$ is the super symmetric product with respect to the $|\bullet$ -grading.

Note that (3.1.6) is tri-graded: it is the direct sum of $S^n L(W_\bullet)_e^d$ spanned by elements of the form

$$\prod_{i=1}^n [x_0^{(i)}, [x_1^{(i)}, \dots, [x_{d_i-1}^{(i)}, x_{d_i}^{(i)}] \dots]]$$

for $x_j^{(i)} \in W_{e_{ij}}$ with $1 \leq i \leq n$, $0 \leq j \leq d_i$ satisfying

$$\sum_{i=1}^n d_i = d, \quad \sum_{i=1}^n \sum_{j=1}^{d_i} e_{ij} = e.$$

Here e is $|\bullet$ -degree, d is $|\bullet$ -degree, and n is called $|\bullet$ -degree. By the $|\bullet$ -graded Leibniz rule, the bracket $[-, -]$ on $L(W_\bullet)$ extends to the $|\bullet$ -graded bracket $\{-, -\}$ on (3.1.6). Then the algebra (3.1.6) is a $|\bullet$ -graded Poisson algebra with respect to the gradings $|\bullet$ and $|\bullet$.

Let A_\bullet be a super commutative graded algebra. By regarding it as a graded vector space, we obtain the $|\bullet$ -graded Poisson algebra $\text{Poiss}(A_\bullet)$. Let $I_{A_\bullet} \subset S(A_\bullet)$ be the ideal given by the exact sequence

$$0 \rightarrow I_{A_\bullet} \rightarrow S(A_\bullet) \xrightarrow{\eta} A_\bullet \rightarrow 0.$$

Here η is given by the multiplication in A_\bullet . The ideal I_{A_\bullet} is generated by elements of the form $a \cdot b - ab$ for $a, b \in A_\bullet$. We then define the ideal

$$(3.1.7) \quad \langle\langle I_{A_\bullet} \rangle\rangle \subset \text{Poiss}(A_\bullet)$$

to be generated by elements of the form

$$\{x_1, \{x_2, \dots, \{x_{k-1}, y\} \dots\}\}, \quad x_1, \dots, x_{k-1} \in A_\bullet, \quad y \in I_{A_\bullet}.$$

Definition 3.3. *The $|\bullet\rangle$ -graded Poisson envelope of A_\bullet is defined by*

$$(3.1.8) \quad P(A_\bullet) := \text{Poiss}(A_\bullet) / \langle\langle I_{A_\bullet} \rangle\rangle.$$

The ideal (3.1.7) is homogeneous with respect to the $|\bullet\rangle$ and $|\bullet\rangle$ grading (but not for \bullet -grading), so the algebra (3.1.8) is $|\bullet\rangle$ -graded: it is the direct sum of $P(A_\bullet)_e^d$, where d is $|\bullet\rangle$ -degree and e is $|\bullet\rangle$ -degree. By the definition of (3.1.7), the Poisson structure on $\text{Poiss}(A_\bullet)$ descends to the $|\bullet\rangle$ -graded Poisson structure on (3.1.8). By forgetting the $|\bullet\rangle$ -grading on (3.1.8), we obtain the $|\bullet\rangle$ -graded Poisson algebra $P(A_\bullet)_\bullet$.

Lemma 3.4. *The functor $A_\bullet \mapsto P(A_\bullet)_\bullet$ is the left adjoint of (3.1.3).*

Proof. The proof is straightforward and left to the reader. \square

Example 3.5. *Let W_\bullet be a finite dimensional graded vector space and set*

$$A_\bullet = S(W_\bullet).$$

Note that A_\bullet is a super commutative graded algebra. In this case, we have the canonical isomorphism of $|\bullet\rangle$ -graded Poisson algebras

$$(3.1.9) \quad P(A_\bullet)_\bullet \xrightarrow{\cong} \text{gr}_F T(W_\bullet)_\bullet.$$

The above isomorphism is proved in [Kap98, Example 4.1.2] when W_\bullet consists of degree zero part, and the same argument works in the general case.

3.2. Description of graded Poisson envelopes. Let A_\bullet be a super commutative graded algebra given by

$$(3.2.1) \quad A_\bullet = R \otimes S(W_\bullet)$$

for a smooth commutative algebra R and a finite dimensional graded vector space W_\bullet . Let Ω_{A_\bullet} be the graded module of differential forms on A_\bullet . Similarly to (3.1.4), let

$$(3.2.2) \quad L_{A_\bullet}(\Omega_{A_\bullet}) \subset \bigoplus_{n \geq 0} \overbrace{\Omega_{A_\bullet} \otimes_{A_\bullet} \dots \otimes_{A_\bullet} \Omega_{A_\bullet}}^n$$

be the super A_\bullet -Lie subalgebra generated by Ω_{A_\bullet} . It has $|\bullet\rangle$ -grading induced by the grading on W_\bullet , and also $|\bullet\rangle$ -grading similarly to (3.1.5). Let $L_{A_\bullet}^+(\Omega_{A_\bullet})$ be the positive degree part of (3.2.2) with respect to the $|\bullet\rangle$ -grading.

Lemma 3.6. *If R is local, we have an isomorphism of $|\bullet\rangle$ -graded Poisson algebras*

$$P(A_\bullet)_\bullet \cong S_{A_\bullet}(L_{A_\bullet}^+(\Omega_{A_\bullet}))_\bullet.$$

Proof. The case of $W_\bullet = 0$ is proved in [Cor04, Theorem 1.4]. The case of $W_\bullet \neq 0$ is similarly proved without any modification. Indeed let $\mathfrak{m} \subset R$ be the maximal ideal of R , and set $V = \mathfrak{m}/\mathfrak{m}^2$. We have the identification

$$(3.2.3) \quad \Omega_{A_\bullet} = A_\bullet \otimes (V \oplus W_\bullet).$$

By the identification (3.2.3), we have

$$S_{A_\bullet}(L_{A_\bullet}^+(\Omega_{A_\bullet})) = A_\bullet \otimes SL^+(V \oplus W_\bullet).$$

The above identification gives a substitute of the third line of the proof of [Cor04, Theorem 1.4], and the rest of the proof is the same. \square

Next we consider the case that R is not necessary local. Let

$$(3.2.4) \quad \pi: SL(A_\bullet)_\bullet \rightarrow P(A_\bullet)_\bullet$$

be the natural projection. The map (3.2.4) vanishes on I_A , hence it factors through the map

$$\bar{\pi}: A_\bullet \otimes SL^+(A_\bullet)_\bullet \twoheadrightarrow P(A_\bullet)_\bullet.$$

The above map preserves both of $|\bullet$ and $|\bullet$ degrees. For $n \geq 0$, let I^n be the ideal of $P(A_\bullet)_\bullet$

$$(3.2.5) \quad I^n := \bar{\pi} \left(A_\bullet \otimes \bigoplus_{p \geq n} SL^+(A_\bullet)_\bullet \right).$$

The ideal I^n is homogeneous in both of $|\bullet$ and $|\bullet$ degrees. We define

$$(3.2.6) \quad GP(A_\bullet)_\bullet := \bigoplus_{n \geq 0} I^n / I^{n+1}.$$

Note that (3.2.6) is a tri-graded algebra: it is a direct sum of

$$(3.2.7) \quad G^n P(A_\bullet)_e^d := (I^n / I^{n+1})_e^d$$

where e is $|\bullet$ -degree and d is $|\bullet$ -degree. The degree n is called $|\bullet$ -degree. Note that (3.2.7) vanishes for $n > d$ so the filtration

$$(3.2.8) \quad P(A_\bullet)_\bullet^d = (I^0)_\bullet^d \supset (I^1)_\bullet^d \supset \cdots \supset (I^n)_\bullet^d \supset \cdots$$

stabilizes for $n > d$.

Lemma 3.7. *There is a natural isomorphism of tri-graded algebras*

$$(3.2.9) \quad S_{A_\bullet}^\bullet L_{A_\bullet}^+(\Omega_{A_\bullet})_\bullet \xrightarrow{\cong} G^\bullet P(A_\bullet)_\bullet.$$

Proof. If W_\bullet consists of degree zero part, then $|\bullet$ -degrees of both sides of (3.2.9) consists of degree zero, and the result is proved in [Cor04, Theorem 1.6]. The case of general $|\bullet$ -graded W_\bullet is similarly proved without any modification. \square

3.3. Graded NC filtration via graded Poisson envelope. Let R be a smooth (not necessary commutative) algebra, and W_\bullet a finite dimensional graded vector space. We consider the graded algebra Λ_\bullet given by (2.3.1), and set

$$A_\bullet := (\Lambda_\bullet)^{ab}.$$

By (2.3.3), the algebra A_\bullet is a super commutative graded algebra of the form (3.2.1). We have the following lemma:

Lemma 3.8. *We have the canonical isomorphism of $|\bullet$ -graded Poisson algebras*

$$P(A_\bullet)_\bullet \xrightarrow{\cong} \text{gr}_F(\Lambda_\bullet)_\bullet.$$

Proof. If $W_\bullet = 0$, the result is the consequence of [Kap98, Theorem 4.2.1], and almost the same proof is applied for $W_\bullet \neq 0$. Note that $\text{gr}_F(\Lambda_\bullet)^0 = A_\bullet$. By the universality of the $|\bullet$ -graded Poisson envelope, we have the canonical morphism of $|\bullet$ -graded Poisson algebras $\phi: P(A_\bullet)_\bullet \rightarrow \text{gr}_F(\Lambda_\bullet)_\bullet$, which also preserves $|\bullet$ -grading. Since both of $P(A_\bullet)_\bullet$ and $\text{gr}_F(\Lambda_\bullet)_\bullet$ are $|\bullet$ -graded A_\bullet -modules, we can interpret them as R^{ab} -modules by the algebra homomorphism

$$R^{ab} \rightarrow A_\bullet, \quad \lambda \mapsto \lambda \otimes 1.$$

It is enough to show that, for any closed point $x \in \operatorname{Spec} R^{ab}$, ϕ induces the isomorphism

$$(3.3.1) \quad P(A_\bullet)_\bullet \otimes_{R^{ab}} R^{ab}/\mathfrak{m}_x \xrightarrow{\cong} \operatorname{gr}_F(\Lambda_\bullet)_\bullet \otimes_{R^{ab}} R^{ab}/\mathfrak{m}_x.$$

Here $\mathfrak{m}_x \subset R^{ab}$ is the maximal ideal which defines x . Let $A_{x,\bullet} := \Lambda_x^{ab} \otimes S(W_\bullet)$. Then we have

$$P(A_\bullet)_\bullet \otimes_{R^{ab}} R^{ab}/\mathfrak{m}_x = P(A_{x,\bullet})_\bullet \otimes_{R^{ab}} R^{ab}/\mathfrak{m}_x.$$

By Lemma 3.6, the RHS is computed as

$$(3.3.2) \quad P(A_{x,\bullet})_\bullet \otimes_{R^{ab}} R^{ab}/\mathfrak{m}_x = S(W_\bullet)_\bullet \otimes SL^+(V \oplus W_\bullet)_\bullet.$$

Here $V = \mathfrak{m}_x/\mathfrak{m}_x^2$. Applying the same argument for $S(V) \otimes S(W_\bullet) = S(V \oplus W_\bullet)$, we obtain

$$(3.3.3) \quad P(S(V \oplus W_\bullet))_\bullet \otimes_{S(V)} S(V)/\mathfrak{m}_0 = S(W_\bullet)_\bullet \otimes SL^+(V \oplus W_\bullet)_\bullet.$$

Here \mathfrak{m}_0 is the maximal ideal of $S(V)$ corresponding to the origin. On the other hand, by [Kap98, Lemma 4.2.2], we have the isomorphism for $j > d$

$$\operatorname{gr}_F(\Lambda_\bullet)_\bullet^d \otimes_{R^{ab}} R^{ab}/\mathfrak{m}_x \xrightarrow{\cong} \operatorname{gr}_F(\Lambda_\bullet/\mathfrak{m}_x^j)_\bullet^d \otimes_{R^{ab}} R^{ab}/\mathfrak{m}_x.$$

Since we have

$$\Lambda_\bullet/\mathfrak{m}_x^j \cong T(V \oplus W_\bullet)/\mathfrak{m}_0^j$$

we have the identification

$$(3.3.4) \quad \operatorname{gr}_F(\Lambda_\bullet)_\bullet \otimes_{R^{ab}} R^{ab}/\mathfrak{m}_x = \operatorname{gr}_F(T(V \oplus W_\bullet))_\bullet \otimes_{S(V)} S(V)/\mathfrak{m}_0.$$

By (3.3.2), (3.3.3), (3.3.4) and the isomorphism (3.1.9), ϕ induces the isomorphism (3.3.1). \square

3.4. NC virtual structure sheaves via perfect obstruction theory. We now return to the situation of Definition 2.14. Similarly to (3.1.4), (3.2.2), for a graded vector bundle $\mathcal{P}_\bullet \rightarrow M$ on a scheme M , we set

$$(3.4.1) \quad L_{\mathcal{O}_M}(\mathcal{P}_\bullet) \subset T_{\mathcal{O}_M}(\mathcal{P}_\bullet)$$

to be the sheaf of super \mathcal{O}_M -Lie algebras generated by \mathcal{P}_\bullet , i.e. each fiber of $L_{\mathcal{O}_M}(\mathcal{P}_\bullet)$ at $x \in M$ is the super Lie algebra $L(\mathcal{P}_\bullet|_x)$. Note that the grading on \mathcal{P}_\bullet induces the $|\bullet$ -grading on (3.4.1). Similarly to (3.1.5), we also have the $|\bullet$ -grading on (3.4.1), and denote by $L_{\mathcal{O}_M}^+(\mathcal{P}_\bullet)$ its positive degree part with respect to the $|\bullet$ -grading. The following is the main result in this section:

Theorem 3.9. *In the situation of Definition 2.14, we have the following formula in $K_0(M)$:*

$$(3.4.2) \quad (\mathcal{O}_M^{\text{ncvir}})^{\leq d} = \mathcal{O}_M^{\text{vir}} \otimes_{\mathcal{O}_M} [S_{\mathcal{O}_M} L_{\mathcal{O}_M}^+(\mathcal{E}_\bullet)^{\leq d}].$$

Here \mathcal{E}_\bullet is the two term complex (2.1.3), and $(-)^{\leq d}$ is the degree $\leq d$ part with respect to the $|\bullet$ -grading.

Proof. For a commutative dg-scheme $(N, \mathcal{O}_{N,\bullet})$, the construction of the graded Poisson envelope yields the sheaf of graded $\mathcal{O}_{N,\bullet}$ -module $P(\mathcal{O}_{N,\bullet})_d^d$ for each $d \in \mathbb{Z}_{\geq 0}$. By Lemma 3.8, we have the isomorphism of graded $\mathcal{O}_{N,\bullet}$ -modules

$$(3.4.3) \quad P(\mathcal{O}_{N,\bullet})_d^{\leq d} \xrightarrow{\cong} \operatorname{gr}_F(\mathcal{O}_{N,\bullet}^{\text{nc}})_d^{\leq d}.$$

Also the construction of the ideals (3.2.5) yields the filtration of graded $\mathcal{O}_{N,\bullet}$ -modules

$$P(\mathcal{O}_{N,\bullet})^{\leq d} = (\mathcal{I}^0)^{\leq d} \supset (\mathcal{I}^1)^{\leq d} \supset \cdots \supset (\mathcal{I}^n)^{\leq d} \supset \cdots$$

which stabilizes due to the stabilization of (3.2.8). Hence using the notation (2.2.3) and (3.2.6), we have the identity in $K_0(\mathcal{O}_{N,\bullet})$

$$(3.4.4) \quad [P(\mathcal{O}_{N,\bullet})^{\leq d}] = [G^\bullet P(\mathcal{O}_{N,\bullet})^{\leq d}].$$

Then by Lemma 3.7, we have the isomorphism of graded $\mathcal{O}_{N,\bullet}$ -modules

$$(3.4.5) \quad S_{\mathcal{O}_{N,\bullet}}^\bullet L_{\mathcal{O}_{N,\bullet}}^+(\Omega_{N,\bullet})^{\leq d} \xrightarrow{\cong} G^\bullet P(\mathcal{O}_{N,\bullet})^{\leq d}.$$

By (3.4.3), (3.4.4), (3.4.5) and Corollary 2.3, we obtain the identity in $K_0(M)$:

$$(\mathcal{O}_M^{\text{ncvir}})^{\leq d} = \mathcal{O}_M^{\text{vir}} \otimes_{\mathcal{O}_M} [S_{\mathcal{O}_M} L_{\mathcal{O}_M}^+(\overline{\Omega}_{N,\bullet}|_M)^{\leq d}].$$

We are left to show the identity

$$(3.4.6) \quad [S_{\mathcal{O}_M} L_{\mathcal{O}_M}^+(\overline{\Omega}_{N,\bullet}|_M)^{\leq d}] = [S_{\mathcal{O}_M} L_{\mathcal{O}_M}^+(\mathcal{E}_\bullet)^{\leq d}].$$

For a partition of n

$$(3.4.7) \quad \lambda = (\lambda_1, \lambda_2, \dots, \lambda_k), \quad \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$$

let V_λ be the corresponding irreducible representation of S_n . Let \mathbf{S}_λ be the Schur functor defined on the category of complexes of vector bundles on M to itself (cf. [PW86, Section 2]):

$$(3.4.8) \quad \mathbf{S}_\lambda : \mathcal{P}_\bullet \mapsto (V_\lambda \otimes_{\mathcal{O}_M} \mathcal{P}_\bullet^{\otimes n})^{S_n}.$$

Since the functor $\mathcal{P}_\bullet \mapsto S_{\mathcal{O}_M} L_{\mathcal{O}_M}^+(\mathcal{P}_\bullet)^{\leq d}$ is a polynomial functor on the category of graded vector bundles on M in the sense of [Mac79, Appendix A], it is described as

$$(3.4.9) \quad \bigoplus_{n \geq 0} \bigoplus_{|\lambda|=n} \mathcal{U}_{\bullet,\lambda} \otimes_{\mathcal{O}_M} \mathbf{S}_\lambda(\mathcal{P}_\bullet)$$

for graded vector bundles $\mathcal{U}_{\bullet,\lambda}$ on M . Here for a partition (3.4.7), we set $|\lambda| = \lambda_1 + \lambda_2 + \cdots + \lambda_k$. Therefore the identity (3.4.6) follows from Lemma 3.10 below. \square

We have used the following lemma, which is probably well-known, but include the proof because of the lack of a reference.

Lemma 3.10. *Let $\mathcal{P}_\bullet, \mathcal{Q}_\bullet$ be bounded complexes of vector bundles on a scheme M and $s : \mathcal{P}_\bullet \rightarrow \mathcal{Q}_\bullet$ a quasi-isomorphism. Then we have the identity $[\mathbf{S}_\lambda(\mathcal{P}_\bullet)] = [\mathbf{S}_\lambda(\mathcal{Q}_\bullet)]$ in $K_0(M)$.*

Proof. By the representation theory of S_n (cf. [FH91, Theorem 6.3]), we have the decomposition of complexes

$$(\mathcal{P}_\bullet)^{\otimes n} = \bigoplus_{|\lambda|=n} \mathbf{S}_\lambda(\mathcal{P}_\bullet) \otimes V_\lambda.$$

Therefore if \mathcal{P}_\bullet is acyclic, then $\mathbf{S}_\lambda(\mathcal{P}_\bullet)$ is also acyclic. Let $\text{Cone}(s)_\bullet$ be the cone of s , which is acyclic as s is quasi-isomorphism. By the above argument, the complex $\mathbf{S}_\lambda(\text{Cone}(s)_\bullet)$ is also acyclic, hence it is zero in $K_0(M)$. On the other hand, as $\text{Cone}(s)_\bullet$ is $\mathcal{Q}_\bullet \oplus \mathcal{P}_\bullet[1]$ as a graded vector bundle, $\mathbf{S}_\lambda(\text{Cone}(s)_\bullet)$ is isomorphic to

$\mathbf{S}_\lambda(\mathcal{Q}_\bullet \oplus \mathcal{P}_\bullet[1])$ as graded vector bundles. We have the decomposition as graded vector bundles (cf. [FH91, Exercise 6.11])

$$(3.4.10) \quad \mathbf{S}_\nu(\mathcal{Q}_\bullet \oplus \mathcal{P}_\bullet[1]) = \bigoplus_{|\lambda|+|\mu|=|\nu|} (\mathbf{S}_\lambda(\mathcal{Q}_\bullet) \otimes_{\mathcal{O}_M} \mathbf{S}_\mu(\mathcal{P}_\bullet[1]))^{\oplus N_{\lambda,\mu}^\nu}.$$

Here $N_{\lambda,\mu}^\nu \in \mathbb{Z}_{\geq 0}$ is determined by the Littlewood-Richardson rule. Hence we obtain the identity in $K_0(M)$:

$$(3.4.11) \quad \sum_{|\lambda|+|\mu|=|\nu|} N_{\lambda,\mu}^\nu [\mathbf{S}_\lambda(\mathcal{Q}_\bullet)] \otimes_{\mathcal{O}_M} [\mathbf{S}_\mu(\mathcal{P}_\bullet[1])] = 0.$$

Applying the above identity to $\text{id}: \mathcal{P}_\bullet \rightarrow \mathcal{P}_\bullet$, we obtain

$$(3.4.12) \quad \sum_{|\lambda|+|\mu|=|\nu|} N_{\lambda,\mu}^\nu [\mathbf{S}_\lambda(\mathcal{P}_\bullet)] \otimes_{\mathcal{O}_M} [\mathbf{S}_\mu(\mathcal{P}_\bullet[1])] = 0.$$

Noting that $N_{\lambda,\emptyset}^\lambda = 1$, the identities (3.4.11), (3.4.12) together with the induction on $|\lambda|$ shows that $[\mathbf{S}_\lambda(\mathcal{P}_\bullet)] = [\mathbf{S}_\lambda(\mathcal{Q}_\bullet)]$. \square

Using Theorem 3.9, we can compute NC virtual structure sheaves in terms of Schur complexes $\mathbf{S}_\lambda(\mathcal{E}_\bullet)$, given by (3.4.8). Note that we have

$$\mathbf{S}_\lambda(\mathcal{E}_\bullet) = \bigwedge^d \mathcal{E}_\bullet, \quad \lambda = \overbrace{(1, 1, \dots, 1)}^d.$$

Corollary 3.11. (i) For $d = 1$, we have the following formula:

$$(\mathcal{O}_M^{\text{ncvir}})^{\leq 1} = \mathcal{O}_M^{\text{vir}} \otimes_{\mathcal{O}_M} (1 + \mathbf{S}_{(1,1)}(\mathcal{E}_\bullet)).$$

(ii) For $d = 2$, we have the following formula:

$$(\mathcal{O}_M^{\text{ncvir}})^{\leq 2} = \mathcal{O}_M^{\text{vir}} \otimes_{\mathcal{O}_M} (1 + \mathbf{S}_{(1,1)}(\mathcal{E}_\bullet) + \mathbf{S}_{(2,1)}(\mathcal{E}_\bullet) + \mathbf{S}_{(2,2)}(\mathcal{E}_\bullet) + \mathbf{S}_{(1,1,1,1)}(\mathcal{E}_\bullet)).$$

Proof. The formula (3.4.2) implies

$$\begin{aligned} (\mathcal{O}_M^{\text{ncvir}})^{\leq 1} &= \mathcal{O}_M^{\text{vir}} \otimes_{\mathcal{O}_M} (1 + L_{\mathcal{O}_M}(\mathcal{E}_\bullet)^1) \\ (\mathcal{O}_M^{\text{ncvir}})^{\leq 2} &= \mathcal{O}_M^{\text{vir}} \otimes_{\mathcal{O}_M} (1 + L_{\mathcal{O}_M}(\mathcal{E}_\bullet)^1 + L_{\mathcal{O}_M}(\mathcal{E}_\bullet)^2 + S_{\mathcal{O}_M}^2 L_{\mathcal{O}_M}(\mathcal{E}_\bullet)^1). \end{aligned}$$

The formula for $d = 1$ then follows from

$$(3.4.13) \quad L_{\mathcal{O}_M}(\mathcal{E}_\bullet)^1 = [\mathcal{E}_\bullet, \mathcal{E}_\bullet] = \bigwedge^2 \mathcal{E}_\bullet = \mathbf{S}_{(1,1)}(\mathcal{E}_\bullet).$$

For $d = 2$, we have the exact sequences of complexes

$$\begin{aligned} 0 \rightarrow \bigwedge^3 \mathcal{E}_\bullet \rightarrow \left(\bigwedge^2 \mathcal{E}_\bullet \right) \otimes \mathcal{E}_\bullet \rightarrow L_{\mathcal{O}_M}(\mathcal{E}_\bullet)^2 \rightarrow 0 \\ 0 \rightarrow \mathbf{S}_{(2,1)}(\mathcal{E}_\bullet) \rightarrow \left(\bigwedge^2 \mathcal{E}_\bullet \right) \otimes \mathcal{E}_\bullet \rightarrow \bigwedge^3 \mathcal{E}_\bullet \rightarrow 0 \end{aligned}$$

showing that $[L_{\mathcal{O}_M}(\mathcal{E}_\bullet)^2] = [\mathbf{S}_{(2,1)}(\mathcal{E}_\bullet)]$. Here the former sequence easily follows from $L_{\mathcal{O}_M}(\mathcal{E}_\bullet)^2 = [\mathcal{E}_\bullet, [\mathcal{E}_\bullet, \mathcal{E}_\bullet]]$ and the latter sequence follows from [FH91, Section 6.1]. Also by [FH91, Exercise 6.16], we have the identity

$$[S_{\mathcal{O}_M}^2 \mathbf{S}_{(1,1)}(\mathcal{E}_\bullet)] = [\mathbf{S}_{(2,2)}(\mathcal{E}_\bullet)] + [\mathbf{S}_{(1,1,1,1)}(\mathcal{E}_\bullet)].$$

By combining these identities, we obtain the desired formula for $d = 2$. \square

Corollary 3.12. *Suppose that M^{vir} has virtual dimension zero. Then we have the identity*

$$(3.4.14) \quad (\mathcal{O}_M^{\text{ncvir}})^{\leq d} = \mathcal{O}_M^{\text{vir}}.$$

Proof. The assumption implies that the complex \mathcal{E}_\bullet given by (2.1.3) is of rank zero, hence any Schur complex $\mathbf{S}_\lambda(\mathcal{E}_\bullet)$ is of rank zero. Since $\mathcal{O}_M^{\text{vir}}$ is written as $[Q] - [Q']$ for zero dimensional sheaves Q, Q' , we obtain the desired identity (3.4.14) by Theorem 3.9. \square

3.5. NC virtual structure sheaves associated to perfect obstruction theory. The result of Theorem 3.9 indicates that one may define the NC virtual structure sheaf from the perfect obstruction theory, without using a quasi NCDG structure. Let M be a scheme and

$$(3.5.1) \quad \phi: \mathcal{E}_\bullet \rightarrow \tau_{\geq -1} \mathbf{L}_M$$

a perfect obstruction theory in the sense of [BF97], i.e. \mathcal{E}_\bullet is a two term complex of vector bundles on M , $\tau_{\geq -1} \mathbf{L}_M$ is the truncated cotangent complex of M , and ϕ is the morphism in the derived category such that $\mathcal{H}_0(\phi)$ is an isomorphism and $\mathcal{H}_{-1}(\phi)$ is surjective. Similarly to (2.2.2), one can define the virtual structure sheaf $\mathcal{O}_M^{\text{vir}} \in K_0(M)$ using data (3.5.1) as pointed out in [BF97, Remark 5.4]. The result of Theorem 3.9 naturally leads to the following definition:

Definition 3.13. *For a perfect obstruction theory (3.5.1) on a scheme M , the d -th NC virtual structure sheaf is defined by*

$$(3.5.2) \quad (\mathcal{O}_M^{\text{ncvir}})^{\leq d} = \mathcal{O}_M^{\text{vir}} \otimes_{\mathcal{O}_M} [S_{\mathcal{O}_M} L_{\mathcal{O}_M}^+(\mathcal{E}_\bullet)^{\leq d}].$$

Example 3.14. *Suppose that M is non-singular, hence $\mathbf{L}_M = \Omega_M$, and the perfect obstruction theory (3.5.1) is given by the identity $\mathcal{E}_\bullet = \Omega_M \rightarrow \Omega_M$. Then we have $\mathcal{O}_M^{\text{vir}} = \mathcal{O}_M$ and*

$$(3.5.3) \quad (\mathcal{O}_M^{\text{ncvir}})^{\leq d} = [S_{\mathcal{O}_M} L_{\mathcal{O}_M}^+(\Omega_M)^{\leq d}].$$

If M admits a d -smooth NC thickening $(M, \mathcal{O}_M^{\leq d})$, then the RHS of (3.5.3) coincides with the K -theory class of $\mathcal{O}_M^{\leq d}$.

Example 3.15. *Suppose that M^{vir} has virtual dimension zero, or equivalently \mathcal{E}_\bullet is rank zero. Similarly to Corollary 3.12, we have the identity*

$$(\mathcal{O}_M^{\text{ncvir}})^{\leq d} = \mathcal{O}_M^{\text{vir}}.$$

In particular if (3.5.1) is a symmetric perfect obstruction theory (cf. [BF08]), then \mathcal{E}_\bullet is rank zero and $(\mathcal{O}_M^{\text{ncvir}})^{\leq d}$ coincides with the commutative virtual structure sheaf.

Example 3.16. *The definition of (3.5.2) also makes sense in the equivariant situation, and gives non-trivial examples of NC virtual structure sheaves with virtual dimension zero. Let $T = (\mathbb{C}^*)^3$ acts on \mathbb{C}^3 by weight $(1, 1, 1)$. By regrading \mathbb{C}^3 as the moduli space of skyscraper sheaves \mathcal{O}_x for $x \in \mathbb{C}^3$, we have the T -equivariant perfect obstruction theory*

$$\Omega_{\mathbb{C}^3} \oplus \bigwedge^2 \Omega_{\mathbb{C}^3}[1] \rightarrow \Omega_{\mathbb{C}^3}.$$

Let (t_1, t_2, t_3) be the T -equivariant parameters. By localization, we obtain the identity in $K_T(\mathbb{C}^3)$

$$\mathcal{O}_{\mathbb{C}^3}^{\text{vir}} = \frac{(1 - t_1^{-1}t_2^{-1})(1 - t_1^{-1}t_3^{-1})(1 - t_2^{-1}t_3^{-1})}{(1 - t_1^{-1})(1 - t_2^{-1})(1 - t_3^{-1})}.$$

By Corollary 3.11 (i), we have the identities in $K_T(\mathbb{C}^3)$

$$\begin{aligned} & (\mathcal{O}_{\mathbb{C}^3}^{\text{ncvir}})^{\leq 1} \\ &= \mathcal{O}_{\mathbb{C}^3}^{\text{vir}} \otimes_{\mathcal{O}_{\mathbb{C}^3}} \left(1 + \bigwedge^2 \Omega_{\mathbb{C}^3} - \Omega_{\mathbb{C}^3} \otimes_{\mathcal{O}_{\mathbb{C}^3}} T_{\mathbb{C}^3} + S_{\mathcal{O}_{\mathbb{C}^3}}^2(T_{\mathbb{C}^3}) \right) \\ &= \frac{(1 - t_1^{-1}t_2^{-1})(1 - t_1^{-1}t_3^{-1})(1 - t_2^{-1}t_3^{-1})}{(1 - t_1^{-1})(1 - t_2^{-1})(1 - t_3^{-1})} \cdot (-2 + t_1^{-1}t_2^{-1} + t_1^{-1}t_3^{-1} + t_2^{-1}t_3^{-1} - t_1^{-1}t_2 \\ &\quad - t_2^{-1}t_3 - t_1^{-1}t_3 - t_1t_2^{-1} - t_2t_3^{-1} - t_1t_3^{-1} + t_1^2 + t_2^2 + t_3^2 + t_1t_2 + t_1t_3 + t_2t_3). \end{aligned}$$

4. CONSTRUCTIONS OF QUASI NCDG STRUCTURES

In the previous section, we introduced the notion of NC virtual structure sheaves (cf. Definition 2.14) of a quasi NC structure, using the notion of a quasi NCDG structure. Although NC virtual structure sheaves turned out to be described using the perfect obstruction theory (cf. Theorem 3.9), still the validity of Definition 2.14 relies on the existence of a quasi NCDG structure. In this section, we show that the moduli spaces of graded modules over graded algebras admit quasi NCDG structures. The results in this section will be used in the next section to show a similar result in a geometric context.

4.1. Graded algebras and quivers. Let A be a graded algebra

$$(4.1.1) \quad A = \bigoplus_{i \geq 0} A_i$$

such that $A_0 = \mathbb{C}$ and each A_i is finite dimensional. We denote by

$$\mathfrak{m} := A_{>0} \subset A$$

the maximal ideal of A . Let $A \text{ mod}_{\text{gr}}$ be the category of finitely generated graded left A -modules. For $M \in A \text{ mod}_{\text{gr}}$, we denote by M_i the degree i -part of M , and write $|a| = i$ for non-zero $a \in M_i$. For $q > p > 0$, we define

$$(4.1.2) \quad A \text{ mod}_{[p,q]} \subset A \text{ mod}_{\text{gr}}$$

to be the subcategory of $M \in A \text{ mod}_{\text{gr}}$ with $M_i = 0$ for $i \notin [p, q]$. The category (4.1.2) is also interpreted as the category of representations of some quiver, defined as follows:

Definition 4.1. For $q > p > 0$, the quiver $Q_{[p,q]}$ is defined as follows: the set of vertices is

$$Q_0 = \{p, p+1, \dots, q\}.$$

The number of arrows in $Q_{[p,q]}$ from i to j is given by $\dim_{\mathbb{C}} \mathfrak{m}_{j-i}$. The set of arrows in $Q_{[p,q]}$ is denoted by Q_1 .

Below we fix bases of \mathfrak{m}_k for each $k \in \mathbb{Z}_{\geq 1}$, and identify the set of arrows from i to j with the set of basis elements of \mathfrak{m}_{j-i} . Let $\mathbb{C}[Q_{[p,q]}]$ be the path algebra of $Q_{[p,q]}$. The multiplication

$$(4.1.3) \quad \vartheta: \mathfrak{m}_{j-i} \otimes \mathfrak{m}_{k-j} \rightarrow \mathfrak{m}_{k-i}$$

in A defines the relation in $Q_{[p,q]}$, by defining the two sided ideal

$$I \subset \mathbb{C}[Q_{[p,q]}]$$

to be generated by elements of the form

$$\vartheta(\alpha \otimes \beta) - \alpha \cdot \beta, \quad \alpha \in \mathfrak{m}_{j-i}, \quad \beta \in \mathfrak{m}_{k-j}.$$

Here we have regarded α, β as formal linear combinations of paths from i to j , j to k , respectively and $\alpha \cdot \beta$ is the multiplication in $\mathbb{C}[Q_{[p,q]}]$.

Definition 4.2. We define $\text{Rep}(Q_{[p,q]})$ to be the category of representations of $Q_{[p,q]}$, i.e. its objects consist of collections

$$(4.1.4) \quad W = (\{W_k\}_{k=p}^q, \{\phi_a\}_{a \in Q_1}), \quad \phi_a: W_{t(a)} \rightarrow W_{h(a)}$$

where W_k is a finite dimensional vector space and ϕ_a is a linear map for each $a \in Q_1$. Here $t(a)$ is the tail of a , and $h(a)$ is the head of a .

For a collection (4.1.4), its *dimension vector* is defined by

$$\dim W := (\dim W_p, \dim W_{p+1}, \dots, \dim W_q) \in \mathbb{Z}_{\geq 0}^{q-p+1}.$$

Given a collection (4.1.4), there is the natural map

$$(4.1.5) \quad \mathbb{C}[Q_{[p,q]}] \rightarrow \text{End}(W_\bullet), \quad W_\bullet := \bigoplus_{k=p}^q W_k$$

sending $a \in Q_1$ to ϕ_a .

Definition 4.3. The subcategory of $(Q_{[p,q]}, I)$ -representations

$$\text{Rep}(Q_{[p,q]}, I) \subset \text{Rep}(Q_{[p,q]})$$

is defined to be the category of collections (4.1.4) such that the map (4.1.5) is zero on I .

By the construction, sending a collection (4.1.4) to W_\bullet gives the equivalence

$$(4.1.6) \quad \text{Rep}(Q_{[p,q]}, I) \xrightarrow{\sim} A \text{ mod}_{[p,q]}.$$

4.2. Constructions of commutative dg-schemes. We are going to construct quasi NCDG structures on the moduli spaces of representations of $Q_{[p,q]}$. Before that, following [BFHR14], we recall the constructions of smooth commutative dg-structures on smooth schemes using the notion of curved DGLA. In this subsection, we assume that N is a commutative smooth scheme.

Definition 4.4. ([BFHR14, Definition 1.2]) A bundle of curved DGLA over N is a graded vector bundle \mathcal{L}_\bullet on N , endowed with the following data:

$$\mu \in \Gamma(\mathcal{L}_2), \quad \delta: \mathcal{L}_\bullet \rightarrow \mathcal{L}_\bullet, \quad [-, -]: \wedge^2 \mathcal{L}_\bullet \rightarrow \mathcal{L}_\bullet$$

where δ is an \mathcal{O}_N -module homomorphism of degree one (called twisted differential), $[-, -]$ is an \mathcal{O}_N -linear super alternating bracket of degree zero, which subject to the following axioms:

- $\delta(\mu) = 0$ as an element of $\Gamma(\mathcal{L}_3)$.
- $\delta \circ \delta = [\mu, -]$.
- δ is a super derivation with respect to the bracket $[-, -]$.
- The bracket $[-, -]$ satisfies the super Jacobi identity.

Given a curved DGLA \mathcal{L}_\bullet over N , we associate a sheaf of super commutative dg-algebras whose underlying \mathcal{O}_N -algebra is

$$\mathcal{O}_{N,\bullet} := S_{\mathcal{O}_N}(\mathcal{L}_\bullet[1]^\vee).$$

By the Leibniz rule, the differential on $\mathcal{O}_{N,\bullet}$ is determined by its restriction to $\mathcal{L}_\bullet[1]^\vee$

$$q = q_0 + q_1 + q_2: \mathcal{L}_\bullet[1]^\vee \rightarrow \mathcal{O}_N \oplus \mathcal{L}_\bullet[1]^\vee \oplus S_{\mathcal{O}_N}^2(\mathcal{L}_\bullet[1]^\vee)$$

where q_0 is given by μ , q_1 is given by δ and q_2 is given by $[-, -]$. The axiom of the curved DGLA shows that $q^2 = 0$, hence we obtain the sheaf of super commutative dg-algebras $(\mathcal{O}_{N,\bullet}, q)$ on N .

We construct a curved DGLA on N using a graded vector bundle

$$\mathcal{V}_\bullet \rightarrow N$$

together with the graded algebra (4.1.1). Note that $\mathcal{E}nd_{\mathcal{O}_N}(\mathcal{V}_\bullet)$ is a graded vector bundle on N whose degree i piece consists of morphisms $\mathcal{V}_\bullet \rightarrow \mathcal{V}_\bullet$ sending \mathcal{V}_j to \mathcal{V}_{j+i} . We define

$$\mathcal{L}_n := \text{Hom}_{\text{gr}}(\mathfrak{m}^{\otimes n}, \mathcal{E}nd_{\mathcal{O}_N}(\mathcal{V}_\bullet)), \quad \mathcal{L}_\bullet := \bigoplus_{n \geq 0} \mathcal{L}_n.$$

Here for graded vector spaces W_1, W_2 , we denote by $\text{Hom}_{\text{gr}}(W_1, W_2)$ the space of linear maps $W_1 \rightarrow W_2$ preserving the degrees. For example, \mathcal{L}_1 is written as

$$\mathcal{L}_1 = \bigoplus_{i \geq 1, j \in \mathbb{Z}} \mathcal{H}om_{\mathcal{O}_N}(\mathfrak{m}_i \otimes \mathcal{V}_j, \mathcal{V}_{j+i}).$$

Note that \mathcal{L}_\bullet is a graded vector bundle on N . We see that \mathcal{L}_\bullet is a sheaf of dg-algebras on N . The differential $d: \mathcal{L}_n \rightarrow \mathcal{L}_{n+1}$ is given by

$$df(a_1 \otimes \cdots \otimes a_{n+1}) = \sum_{i=1}^n (-1)^{n-i} f(a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+1}).$$

The composition $\circ: \mathcal{L}_m \times \mathcal{L}_n \rightarrow \mathcal{L}_{m+n}$ is given by

$$f \circ f'(a_1 \otimes \cdots \otimes a_{m+n}) = (-1)^{mn} f(a_1 \otimes \cdots \otimes a_m) \circ f'(a_{m+1} \otimes \cdots \otimes a_{m+n}).$$

It is easy to check that the triple

$$(4.2.1) \quad (\mathcal{L}_\bullet, d, \circ)$$

determines the sheaf of dg-algebras on N .

Now suppose that e is a section of $\mathcal{L}_1 \rightarrow N$, i.e. e is a degree preserving linear map

$$e: \mathfrak{m} \rightarrow \text{End}_{\mathcal{O}_N}(\mathcal{V}_\bullet).$$

We will construct a curved DGLA associated to the above data, with the underlying graded vector bundle is

$$\mathcal{L}_{\geq 2} := \bigoplus_{n \geq 2} \mathcal{L}_n.$$

The element $\mu \in \Gamma(\mathcal{L}_2)$ is defined by

$$\mu := de + e \circ e, \quad \mu(a_1 \otimes a_2) = e(a_1 a_2) - e(a_1) \circ e(a_2).$$

The bracket is

$$[f, f'] = f \circ f' - (-1)^{mn} f' \circ f$$

for $f \in \mathcal{L}_n, f' \in \mathcal{L}_m$. The twisted differential is defined by

$$\delta := d + [e, -]: \mathcal{L}_{\geq 2} \rightarrow \mathcal{L}_{\geq 2}.$$

It is easy to see that the triple $(\mathcal{L}_{\geq 2}, \delta, [-, -])$ is a curved DGLA, hence determines the commutative dg-scheme

$$(4.2.2) \quad (N, S_{\mathcal{O}_N}(\mathcal{L}_{\geq 2}[1]^\vee)) = \left(N, S_{\mathcal{O}_N} \left(\bigoplus_{n \geq 2} \text{Hom}_{\text{gr}}(\mathfrak{m}^{\otimes n}, \mathcal{E}nd_{\mathcal{O}_N}(\mathcal{V}_\bullet)) [1]^\vee \right) \right).$$

The zero-th truncation of the above dg-scheme is the scheme theoretic zero locus of the section μ .

4.3. (DG) moduli spaces of graded modules. Let us fix $q > p > 0$ and non-negative integers

$$\gamma = (\gamma_p, \gamma_{p+1}, \dots, \gamma_q) \in \mathbb{Z}_{\geq 0}^{q-p+1}.$$

Let W_\bullet be a finite dimensional graded vector space written as

$$W_\bullet = \bigoplus_{k=p}^q W_k, \quad \dim W_k = \gamma_k.$$

Then W_\bullet is a graded vector bundle on a point, hence the construction of the previous subsection yields the dg-algebra

$$L := \bigoplus_{n \geq 0} L_n, \quad L_n := \text{Hom}_{\text{gr}}(\mathfrak{m}^{\otimes n}, \text{End}(W_\bullet)).$$

We define the following scheme theoretic Maurer-Cartan locus

$$(4.3.1) \quad MC(L) := \{x \in L^1 : dx + x \circ x = 0\}.$$

Note that an element

$$x \in L^1 = \text{Hom}_{\text{gr}}(\mathfrak{m}, \text{End}(W_\bullet))$$

corresponds to a representation of $Q_{[p,q]}$, and it is contained in $MC(L)$ if and only if it corresponds to an object in the subcategory $\text{Rep}(Q_{[p,q]}, I) \subset \text{Rep}(Q_{[p,q]})$. We next consider the stability condition on $\text{Rep}(Q_{[p,q]})$.

Definition 4.5. *An object $W \in \text{Rep}(Q_{[p,q]})$ is called (semi)stable if for any sub object $0 \neq W' \subsetneq W$ in $\text{Rep}(Q_{[p,q]})$, we have the inequality*

$$\dim W_p \cdot \dim W'_q > (\geq) \dim W_q \cdot \dim W'_p.$$

We have the Cartesian square

$$(4.3.2) \quad \begin{array}{ccc} MC(L)^s & \hookrightarrow & (L^1)^s \\ \downarrow & & \downarrow \\ MC(L) & \hookrightarrow & L^1. \end{array}$$

Here $MC(L)^s$, $(L^1)^s$ correspond to stable objects in $\text{Rep}(Q_{[p,q]}, I)$, $\text{Rep}(Q_{[p,q]})$ respectively. The vertical inclusions in (4.3.2) are open immersions and the horizontal inclusions are closed embeddings. Let G be the group of degree preserving linear isomorphisms $W_\bullet \rightarrow W_\bullet$, i.e.

$$G := \prod_{k=p}^q \text{GL}(W_k).$$

Then L admits the action of G by

$$(g \cdot f)(a_1 \otimes \cdots \otimes a_n) = g \circ f(a_1 \otimes \cdots \otimes a_n) \circ g^{-1}$$

where $f \in L_n$ and $g \in G$. The dg-algebra structure on L is G -equivariant, hence the diagram (4.3.2) is also G -equivariant. Since the automorphisms of stable representations are \mathbb{C}^* , the stabilizer group of the G -action on $(L^1)^s$ is the diagonal subgroup $\mathbb{C}^* \subset G$, hence the G -action on $(L^1)^s$ descends to the free action of the quotient group $\overline{G} = G/\mathbb{C}^*$. The free quotients

$$(4.3.3) \quad M_\gamma := MC^s(L)/\overline{G}, \quad N_\gamma := (L^1)^s/\overline{G}$$

are indeed obtained as GIT quotients (cf. [Kin94]), hence they are quasi projective schemes. By [Kin94], the scheme N_γ is the coarse moduli space of stable $Q_{[p,q]}$ -representations, and M_γ is the closed subscheme of N_γ corresponding to stable $(Q_{[p,q]}, I)$ -representations. Note that N_γ is non-singular, since it is a free quotient of a smooth variety.

Now suppose that γ is a primitive dimension vector, i.e.

$$\text{g.c.d.}\{\gamma_i : p \leq i \leq q\} = 1.$$

Then by [Kin94], N_γ admits a universal representation

$$(4.3.4) \quad \mathcal{V} = (\{\mathcal{V}_i\}_{i=p}^q, \{\phi_a\}_{a \in Q_1}), \quad \phi_a : \mathcal{V}_{t(a)} \rightarrow \mathcal{V}_{h(a)}$$

i.e. each \mathcal{V}_i is a vector bundle on N_γ , ϕ_a is a morphism of vector bundles such that for any $x \in N_\gamma$, the restriction $\mathcal{V}|_x$ is the representation of $Q_{[p,q]}$ corresponding to x . Note that

$$\mathcal{V}_\bullet = \bigoplus_{i=p}^q \mathcal{V}_i \rightarrow N_\gamma$$

is a graded vector bundle, and the collection of morphisms ϕ_a corresponds to the graded preserving linear map

$$(4.3.5) \quad e : \mathfrak{m} \rightarrow \text{End}_{\mathcal{O}_{N_\gamma}}(\mathcal{V}_\bullet).$$

Therefore the construction of the dg-scheme (4.2.2) yields the commutative dg-structure on N_γ

$$(4.3.6) \quad (N_\gamma, \mathcal{O}_{N_\gamma, \bullet}), \quad \mathcal{O}_{N_\gamma, \bullet} = S_{\mathcal{O}_{N_\gamma}} \left(\bigoplus_{n \geq 2} \text{Hom}_{\text{gr}}(\mathfrak{m}^{\otimes n} \otimes \mathcal{V}_\bullet, \mathcal{V}_\bullet)[1]^\vee \right).$$

By (4.3.1) and the construction of M_γ , the zero-th truncation of $(N_\gamma, \mathcal{O}_{N_\gamma, \bullet})$ coincides with the closed subscheme $M_\gamma \subset N_\gamma$.

4.4. Quasi NC structures on M_γ . As before, we assume that γ is a primitive dimension vector of $Q_{[p,q]}$, so that there exists a universal $Q_{[p,q]}$ -representation (4.3.4). Let $U \subset N_\gamma$ be an affine open subset such that each $\mathcal{V}_k|_U$ is a trivial vector bundle

$$\mathcal{V}_k = \mathcal{O}_U \otimes W_k, \quad p \leq k \leq q$$

where W_k is a vector space with dimension γ_k . Since U is a smooth affine scheme, there is an NC smooth thickening (cf. [Kap98, Theorem 1.6.1])

$$U^{\text{nc}} = (U, \mathcal{O}_U^{\text{nc}})$$

on U , which is unique up to non-canonical isomorphisms. We set

$$\mathcal{V}_{U,k}^{\text{nc}} := \mathcal{O}_U^{\text{nc}} \otimes W_k, \quad p \leq k \leq q$$

and regard them as left $\mathcal{O}_U^{\text{nc}}$ -modules. Since $\mathcal{O}_U^{\text{nc}} \twoheadrightarrow \mathcal{O}_U$ is surjective, each morphism ϕ_a lifts to a left $\mathcal{O}_U^{\text{nc}}$ -module homomorphism

$$(4.4.1) \quad \phi_a^{\text{nc}} : \mathcal{V}_{t(a)}^{\text{nc}} \rightarrow \mathcal{V}_{h(a)}^{\text{nc}}, \quad a \in Q_1.$$

Then the data

$$(4.4.2) \quad \mathcal{V}_U^{\text{nc}} := (\{\mathcal{V}_{U,k}^{\text{nc}}\}_{k=p}^q, \{\phi_a^{\text{nc}}\}_{a \in Q_1})$$

is a flat family of representations of $Q_{[p,q]}$ over the NC scheme U^{nc} . Here we refer to [Todb, Definition 3.1] for the definition of flat representations of quivers over NC schemes. As before, we set

$$\mathcal{V}_{U,\bullet}^{\text{nc}} := \bigoplus_{k=p}^q \mathcal{V}_{U,k}^{\text{nc}}.$$

The two sided ideal $\mathcal{I}_{U,I} \subset \mathcal{O}_U^{\text{nc}}$ is defined by the image of

$$(4.4.3) \quad \mathcal{V}_{U,\bullet}^{\text{nc}} \otimes I \otimes (\mathcal{V}_{U,\bullet}^{\text{nc}})^\vee \rightarrow \mathcal{V}_{U,\bullet}^{\text{nc}} \otimes \text{End}_{\mathcal{O}_U^{\text{nc}}}(\mathcal{V}_{U,\bullet}^{\text{nc}}) \otimes (\mathcal{V}_{U,\bullet}^{\text{nc}})^\vee \rightarrow \mathcal{O}_U^{\text{nc}}.$$

Here the first map of (4.4.3) is induced by

$$I \subset \mathbb{C}[Q_{[p,q]}] \rightarrow \text{End}_{\mathcal{O}_U^{\text{nc}}}(\mathcal{V}_{U,\bullet}^{\text{nc}})$$

sending $a \in Q_1$ to ϕ_a^{nc} , and the second map of (4.4.3) is given by $x \otimes g \otimes f \mapsto f \circ g(x)$. We set

$$(4.4.4) \quad V := M_\gamma \cap U, \quad \mathcal{O}_V^{\text{nc}} := \mathcal{O}_U^{\text{nc}} / \overline{\mathcal{I}_{U,I}}, \quad V^{\text{nc}} := (V, \mathcal{O}_V^{\text{nc}}).$$

Here $\overline{\mathcal{I}_{U,I}}$ is the topological closure of $\mathcal{I}_{U,I}$ with respect to the NC filtration of $\mathcal{O}_U^{\text{nc}}$. Then V^{nc} is an NC structure on V .

Note that giving a collection of morphisms (4.4.1) is equivalent to giving an element

$$(4.4.5) \quad \widehat{e} \in \text{Hom}_{\text{gr}}(\mathfrak{m}, \text{End}_{\mathcal{O}_U^{\text{nc}}}(\mathcal{V}_{U,\bullet}^{\text{nc}}))$$

such that $\widehat{e}^{ab} = e|_U$, where e is the universal map (4.3.5). Similarly to the construction (4.2.1), the direct sum

$$\bigoplus_{n \geq 0} \text{Hom}_{\text{gr}}(\mathfrak{m}^{\otimes n}, \text{End}_{\mathcal{O}_U^{\text{nc}}}(\mathcal{V}_{U,\bullet}^{\text{nc}}))$$

is a dg-algebra. We set

$$\widehat{\mu} := d\widehat{e} + \widehat{e} \circ \widehat{e} \in \text{Hom}_{\text{gr}}(\mathfrak{m}^{\otimes 2}, \text{End}_{\mathcal{O}_U^{\text{nc}}}(\mathcal{V}_{U,\bullet}^{\text{nc}})).$$

Then we have the natural morphism of $\mathcal{O}_U^{\text{nc}}$ bi-module

$$(4.4.6) \quad (\mathcal{V}_{U,\bullet}^{\text{nc}} \otimes \mathfrak{m}^{\otimes 2} \otimes (\mathcal{V}_{U,\bullet}^{\text{nc}})^\vee)_0 \rightarrow \mathcal{O}_U^{\text{nc}}.$$

Here $(-)_0$ means the degree zero part, and the map (4.4.6) is given by

$$x \otimes a_1 \otimes a_2 \otimes f \mapsto f \circ \widehat{\mu}(a_1 \otimes a_2)(x).$$

From the construction, it is easy to see that the image of (4.4.6) coincides with $\mathcal{J}_{U,I} \subset \mathcal{O}_U^{\text{nc}}$.

We can give a moduli theoretic interpretation of the NC thickening V^{nc} of V . Let \mathcal{N} be the category of NC nilpotent algebras and

$$h_\gamma|_V : \mathcal{N} \rightarrow \text{Set}$$

the functor sending R to the isomorphism classes of triples (f, \mathcal{W}, ψ) :

- f is a morphism of schemes $f : \text{Spec } R^{ab} \rightarrow V$.
- \mathcal{W} is a flat representation of $(Q_{[p,q]}, I)$ over $\text{Spf } R$.
- ψ is an isomorphism $\psi : \mathcal{W}^{ab} \xrightarrow{\cong} f^* \mathcal{V}$ as $(Q_{[p,q]}, I)$ -representations over $\text{Spec } R^{ab}$.

An isomorphism $(f, \mathcal{W}, \psi) \rightarrow (f', \mathcal{W}', \psi')$ exists if $f = f'$, and there is an isomorphism $\mathcal{W} \rightarrow \mathcal{W}'$ as representations of $(Q_{[p,q]}, I)$ over $\text{Spf } R$ commuting ψ, ψ' .

Proposition 4.6. ([Todb, Proposition 3.11]) *The natural transformation*

$$(4.4.7) \quad h_{V^{\text{nc}}} := \text{Hom}(\text{Spf}(-), V^{\text{nc}}) \rightarrow h_\gamma|_V$$

sending $g : \text{Spf } R \rightarrow V^{\text{nc}}$ to $(g^{ab}, g^* \mathcal{V}_U^{\text{nc}}, id)$ is an NC hull of $h_\gamma|_V$, i.e. (4.4.7) is an isomorphism on the category of commutative algebras, and for any central extension (2.4.1) in \mathcal{N} , we have the surjection:

$$h_{V^{\text{nc}}}(R_1) \twoheadrightarrow h_\gamma|_V(R_1) \times_{h_\gamma|_V(R_2)} h_{V^{\text{nc}}}(R_2).$$

Let $\{U_i\}_{i \in \mathbb{I}}$ be an affine open cover of N_γ such that each $\mathcal{V}_k|_{U_i}$ is trivial, and set $V_i := M_\gamma \cap U_i$. Applying the construction (4.4.4), we obtain affine NC structures on each V_i

$$V_i^{\text{nc}} = (V_i, \mathcal{O}_{V_i}^{\text{nc}}), \quad i \in \mathbb{I}.$$

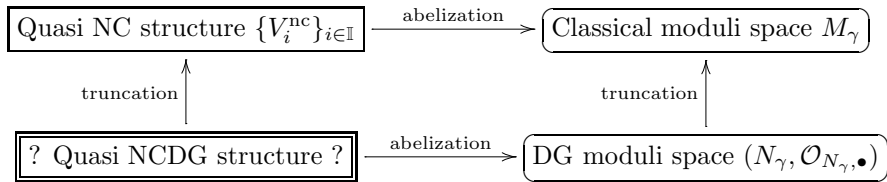
Using Proposition 4.6, we proved the following in [Todb]:

Theorem 4.7. ([Todb, Corollary 3.12]) *There exist isomorphisms*

$$\phi_{ij} : V_j^{\text{nc}}|_{V_{ij}} \xrightarrow{\cong} V_i^{\text{nc}}|_{V_{ij}}, \quad g_{ij} : \phi_{ij}^* \mathcal{V}_{U_i}^{\text{nc}}|_{V_{ij}} \xrightarrow{\cong} \mathcal{V}_{U_j}^{\text{nc}}|_{V_{ij}}$$

where ϕ_{ij} are isomorphisms of NC schemes giving a quasi NC structure on M_γ , and g_{ij} are isomorphisms of representations of $(Q_{[p,q]}, I)$ over $V_j^{\text{nc}}|_{V_{ij}}$.

The constructions of this subsection and the previous subsection are summarized by the following diagram:



Below, we are going to construct a quasi NCDG structure which fits into the above diagram.

4.5. Constructions of non-commutative dg-algebras. Let A be a graded algebra (4.1.1), and R another associative (not necessary commutative) algebra. Let P be a graded free right R -module, and set $P^\vee := \text{Hom}_R(P, R)$ which is a graded free left R -module. We set

$$\mathfrak{P} := \bigoplus_{n \geq 2} (P^\vee \otimes \mathfrak{m}^{\otimes n} \otimes P)_0$$

which is a free graded R bi-module. Here the grading of \mathfrak{P} on $(P^\vee \otimes \mathfrak{m}^{\otimes n} \otimes P)_0$ is set to be $1 - n$. We define the graded algebra \mathfrak{A} to be the tensor algebra of \mathfrak{P} over R

$$\mathfrak{A} := \bigoplus_{m \geq 0} \mathfrak{P}^{\otimes_R m}, \quad \mathfrak{P}^{\otimes_R m} := \overbrace{\mathfrak{P} \otimes_R \mathfrak{P} \otimes_R \cdots \otimes_R \mathfrak{P}}^m.$$

The grading on \mathfrak{A} is induced by that of \mathfrak{P} , and the degree zero part of \mathfrak{A} is $\mathfrak{P}^{\otimes_R 0} := R$. The algebra structure on \mathfrak{A} is given by

$$(b_1 \otimes \cdots \otimes b_m) \cdot (b_{m+1} \otimes \cdots \otimes b_n) = b_1 \otimes \cdots \otimes b_m \otimes b_{m+1} \otimes \cdots \otimes b_n.$$

Let

$$(4.5.1) \quad \widehat{e}: \mathfrak{m} \rightarrow \text{End}_R(P)$$

be a grade preserving linear map. For $a \in \mathfrak{m}$, $x \in P$ and $f \in P^\vee$, we set

$$(4.5.2) \quad ax := \widehat{e}(a)(x) \in P, \quad fa := f \circ \widehat{e}(a) \in P^\vee.$$

We also set the linear map

$$\widehat{\mu}: \mathfrak{m}^{\otimes 2} \rightarrow \text{End}_R(P)$$

as follows:

$$\widehat{\mu}(a_1 \otimes a_2) = \widehat{e}(a_1 a_2) - \widehat{e}(a_1) \circ \widehat{e}(a_2).$$

We define the degree one R bi-module map

$$(4.5.3) \quad Q = Q_0 + Q_1 + Q_2: \mathfrak{P} \rightarrow R \oplus \mathfrak{P} \oplus (\mathfrak{P} \otimes_R \mathfrak{P})$$

in the following way. The map Q_0 is defined by

$$(4.5.4) \quad \begin{aligned} Q_0: (P^\vee \otimes \mathfrak{m}^{\otimes 2} \otimes P)_0 &\rightarrow R \\ f \otimes a_1 \otimes a_2 \otimes x &\mapsto f \circ \widehat{\mu}(a_1, a_2)(x). \end{aligned}$$

The map Q_1 is defined by

$$\begin{aligned} Q_1: (P^\vee \otimes \mathfrak{m}^{\otimes n} \otimes P)_0 &\rightarrow (P^\vee \otimes \mathfrak{m}^{\otimes n-1} \otimes P)_0 \\ f \otimes a_1 \otimes \cdots \otimes a_n \otimes x &\mapsto (-1)^{n+1} f a_1 \otimes a_2 \otimes \cdots \otimes a_n \otimes x \\ &\quad + \sum_{j=1}^{n-1} (-1)^{n+1-j} f \otimes a_1 \otimes \cdots \otimes a_j a_{j+1} \otimes \cdots \otimes a_n \otimes x \\ &\quad - f \otimes a_1 \otimes \cdots \otimes a_{n-1} \otimes a_n x. \end{aligned}$$

Here we have used the convention in (4.5.2). Finally the map Q_2 is defined by

$$Q_2: (P^\vee \otimes \mathfrak{m}^{\otimes n} \otimes P)_0 \rightarrow \bigoplus_{k=2}^{n-2} (P^\vee \otimes \mathfrak{m}^{\otimes k} \otimes P)_0 \otimes_R (P^\vee \otimes \mathfrak{m}^{\otimes n-k} \otimes P)_0$$

$$f \otimes a_1 \otimes \cdots \otimes a_n \otimes x \mapsto \sum_{k=2}^{n-2} (-1)^{n(k-2)+1} f \otimes a_1 \otimes \cdots \otimes a_k \otimes \widehat{\text{id}}_P$$

$$\otimes a_{k+1} \otimes \cdots \otimes a_n \otimes x.$$

Here $\widehat{\text{id}}_P$ is defined as follows: we decompose $\text{id}_P \in \text{Hom}_R(P, P) = P \otimes_R P^\vee$ as

$$\text{id}_P = \sum_i u_i \otimes_R v_i$$

for homogeneous elements $u_i \in P$, $v_i \in P^\vee$ with $|u_i| + |v_i| = 0$, and

$$\widehat{\text{id}}_P := \sum_{|f|+|a_1|+\cdots+|a_k|+|u_i|=0} u_i \otimes_R v_i.$$

By the Leibniz rule, the map (4.5.3) extends to the degree one R bi-module map

$$(4.5.5) \quad Q: \mathfrak{A} \rightarrow \mathfrak{A}.$$

We have the following proposition:

Proposition 4.8. *The map Q in (4.5.5) satisfies $Q^2 = 0$. Hence (\mathfrak{A}, Q) is a non-commutative differential graded algebra.*

Proof. It is straightforward to check $Q^2 = 0$, and we leave the details to the readers. \square

The first few terms of the complex (\mathfrak{A}, Q) is

$$\cdots \rightarrow (P^\vee \otimes \mathfrak{m}^{\otimes 3} \otimes P)_0 \oplus (P^\vee \otimes \mathfrak{m}^{\otimes 2} \otimes P)_0^{\otimes_{R^2}}$$

$$\rightarrow (P^\vee \otimes \mathfrak{m}^{\otimes 2} \otimes P)_0 \xrightarrow{Q_0} R \rightarrow 0.$$

In particular, we have

$$(4.5.6) \quad \mathcal{H}_0(\mathfrak{A}, Q) = R/J$$

where J is the two sided ideal given by the image of Q_0 in (4.5.4).

4.6. Abelianization of \mathfrak{A} . We describe the abelianization of the non-commutative dg-algebra \mathfrak{A} . We set

$$\mathfrak{P}^{ab} := \bigoplus_{n \geq 2} (\mathfrak{m}^{\otimes n} \otimes \text{End}_{R^{ab}}(P^{ab}))_0$$

$$= \bigoplus_{n \geq 2} \text{Hom}_{\text{gr}}(\mathfrak{m}^{\otimes n}, \text{End}_{R^{ab}}(P^{ab}))^\vee.$$

which is a graded free R^{ab} -module. The grading on $(\mathfrak{m}^{\otimes n} \otimes \text{End}_{R^{ab}}(P^{ab}))_0$ is $1-n$, and ${}^\vee$ is the dual of $*$ over R^{ab} .

Lemma 4.9. *As a graded algebra, we have*

$$(4.6.1) \quad \mathfrak{A}^{ab} = S_{R^{ab}}(\mathfrak{P}^{ab}).$$

Proof. We write P as $P = W \otimes R$ for a graded vector space W , and set

$$\overline{W} := \bigoplus_{n \geq 2} (W^\vee \otimes \mathfrak{m}^{\otimes n} \otimes W)_0.$$

Then we have $\mathfrak{P} = R \otimes \overline{W} \otimes R$, and

$$\mathfrak{A} = R * T(\overline{W}).$$

On the other hand, we have $\mathfrak{P}^{ab} = R^{ab} \otimes \overline{W}$, hence

$$S_{R^{ab}}(\mathfrak{P}^{ab}) = R^{ab} \otimes S(\overline{W}).$$

Therefore we obtain (4.6.1). \square

By the Leibniz rule, the derivation (4.5.5) induces a degree one R^{ab} -linear derivation

$$q := Q^{ab}: \mathfrak{A}^{ab} \rightarrow \mathfrak{A}^{ab}.$$

From the description of Q , it is easy to describe q under the identity (4.6.1). By Lemma 4.9, the map q is determined by its restriction to \mathfrak{P}^{ab}

$$q = q_0 + q_1 + q_2: \mathfrak{P}^{ab} \rightarrow R^{ab} \oplus \mathfrak{P}^{ab} \oplus S_{R^{ab}}^2(\mathfrak{P}^{ab}).$$

Let e, μ be the compositions of $\widehat{e}, \widehat{\mu}$, with the natural map $\text{End}_R(P) \rightarrow \text{End}_{R^{ab}}(P^{ab})$:

$$(4.6.2) \quad e: \mathfrak{m} \rightarrow \text{End}_{R^{ab}}(P^{ab}), \quad \mu: \mathfrak{m}^{\otimes 2} \rightarrow \text{End}_{R^{ab}}(P^{ab}).$$

The map q_0 is described as

$$\begin{aligned} q_0: (\mathfrak{m}^{\otimes 2} \otimes \text{End}_{R^{ab}}(P^{ab}))_0 &\rightarrow R^{ab} \\ a_1 \otimes a_2 \otimes g &\mapsto \text{tr}(g \circ \mu(a_1, a_2)). \end{aligned}$$

The map q_1 is described as

$$\begin{aligned} q_1: (\mathfrak{m}^{\otimes n} \otimes \text{End}_{R^{ab}}(P^{ab}))_0 &\rightarrow (\mathfrak{m}^{\otimes n-1} \otimes \text{End}_{R^{ab}}(P^{ab}))_0 \\ a_1 \otimes \cdots \otimes a_n \otimes g &\mapsto (-1)^{n+1} a_2 \otimes \cdots \otimes a_n \otimes (g \circ e(a_1)) \\ &\quad + \sum_{j=1}^{n-1} (-1)^{n+1-j} a_1 \otimes \cdots \otimes a_j a_{j+1} \otimes \cdots \otimes a_n \otimes g \\ &\quad - a_1 \otimes \cdots \otimes a_{n-1} \otimes (e(a_n) \circ g). \end{aligned}$$

The map q_2 is described as

$$\begin{aligned} q_2: (\mathfrak{m}^{\otimes n} \otimes \text{End}_{R^{ab}}(P^{ab}))_0 &\rightarrow \bigoplus_{k=2}^{n-2} (\mathfrak{m}^{\otimes k} \otimes \text{End}_{R^{ab}}(P^{ab}))_0 \otimes_{R^{ab}} (\text{End}_{R^{ab}}(P^{ab}) \otimes \mathfrak{m}^{\otimes n-k})_0 \\ a_1 \otimes \cdots \otimes a_n \otimes g &\mapsto \sum_{k=2}^{n-2} (-1)^{n(k-2)+1} a_1 \otimes \cdots \otimes a_k \otimes \widehat{\circ}^\vee g \otimes a_{k+1} \otimes \cdots \otimes a_n. \end{aligned}$$

Here \circ^\vee is the dual of the composition map

$$\circ^\vee: \text{End}_{R^{ab}}(P^{ab}) \rightarrow \text{End}_{R^{ab}}(P^{ab}) \otimes_{R^{ab}} \text{End}_{R^{ab}}(P^{ab})$$

and writing $\circ^\vee g$ as the sum of $u_i \otimes_{R^{ab}} v_i$ for homogeneous elements $u_i, v_i \in \text{End}_{R^{ab}}(P^{ab})$, we set

$$\hat{\circ}^\vee g = \sum_{|a_1|+\dots+|a_k|+|u_i|=0} u_i \otimes_{R^{ab}} v_i.$$

On the other hand, note that

$$\tilde{P}^{ab} \rightarrow \text{Spec } R^{ab}$$

is a graded vector bundle on $\text{Spec } R^{ab}$. The data of e in (4.6.2) together with the construction of (4.2.2) yield the affine commutative dg-scheme

$$(4.6.3) \quad \left(\text{Spec } R^{ab}, S_{\tilde{R}^{ab}} \left(\bigoplus_{n \geq 2} \text{Hom}_{\text{gr}}(\mathfrak{m}^{\otimes n}, \mathcal{E}nd_{\tilde{R}^{ab}}(\tilde{P}^{ab}))[1]^\vee \right) \right).$$

By Lemma 4.9 together with the above description of $q = Q^{ab}$, the global section of the dg-structure sheaf of (4.6.3) coincides with \mathfrak{A}^{ab} as a dg-algebra.

4.7. Quasi NCDG structures on N_γ . Now we return to the situation of Subsection 4.4. As in Subsection 4.4, we take an affine open subset $U \subset N_\gamma$ such that each $\mathcal{V}_k|_U$ is trivial $\mathcal{V}_k|_U = \mathcal{O}_U \otimes W_k$. We take an NC smooth thickening U^{nc} of U , and a lift $\mathcal{V}_{U,\bullet}^{\text{nc}}$ of $\mathcal{V}|_U$ to a flat representation of $Q_{[p,q]}$ over U^{nc} , as in (4.4.2). We apply the construction in Subsection 4.5 by setting

$$R = \Gamma(\mathcal{O}_U^{\text{nc}}), \quad P = \Gamma(\mathcal{V}_{U,\bullet}^{\text{nc}})^\vee$$

where $^\vee$ is the dual of $*$ over R . Note that P is a graded free right R -module. Using (4.4.5) instead of (4.5.1), the construction in Proposition 4.8 yields the non-commutative dg-algebra structure on

$$(4.7.1) \quad \Lambda_{U,\bullet}^{\text{nc}} := \bigoplus_{m \geq 0} \left(\bigoplus_{n \geq 2} \Gamma(\mathcal{V}_{U,\bullet}^{\text{nc}}) \otimes \mathfrak{m}^{\otimes n} \otimes \Gamma(\mathcal{V}_{U,\bullet}^{\text{nc}})^\vee \right)_0^{\otimes_{\mathcal{O}_U^{\text{nc}}} m}.$$

By the proof of Lemma 4.9, we have

$$(4.7.2) \quad \Lambda_{U,\bullet}^{\text{nc}} = \Gamma(\mathcal{O}_U^{\text{nc}}) * T(\overline{W})$$

where \overline{W} is the finite dimensional graded vector space given by

$$\overline{W} = \bigoplus_{n \geq 2} \left(\bigoplus_{p \leq j, k \leq q} W_j \otimes \mathfrak{m}^{\otimes n} \otimes W_k^\vee \right)_0.$$

Here W_j is a vector space with dimension γ_j , located in degree j . We define the following affine NCDG scheme

$$(4.7.3) \quad (U, \mathcal{O}_{U,\bullet}^{\text{nc}}) := \text{Spf } \Lambda_{U,\bullet}^{\text{nc}}.$$

Note that (4.7.3) is smooth as U^{nc} is a smooth NC thickening of U . By the identity (4.5.6), we have

$$(4.7.4) \quad \tau_0(U, \mathcal{O}_{U,\bullet}^{\text{nc}}) = (V, \mathcal{O}_V^{\text{nc}})$$

where $\mathcal{O}_V^{\text{nc}}$ is given in (4.4.4), as it is given by the NC completion of the cokernel of (4.4.6). Also the argument in the previous subsection shows that

$$(4.7.5) \quad (\mathcal{O}_{U,\bullet}^{\text{nc}})^{ab} = \mathcal{O}_{N_\gamma,\bullet}|_U$$

where $\mathcal{O}_{N_\gamma, \bullet}$ is the sheaf of commutative dg-algebras on N_γ given in (4.3.6). Hence (4.7.3) is an affine NCDG structure on $(U, \mathcal{O}_{N_\gamma, \bullet}|_U)$.

Let $\{U_i\}_{i \in \mathbb{I}}$ be an affine open cover of N_γ , such that each $\mathcal{V}_k|_{U_i}$ is trivial. Applying the above construction, we obtain affine NCDG schemes

$$(U_i, \mathcal{O}_{U_i, \bullet}^{\text{nc}}), \quad i \in \mathbb{I}.$$

On the other hand, we have isomorphisms of NC schemes and $Q_{[p,q]}$ -representations over $U_j^{\text{nc}}|_{U_{ij}}$ (cf. [Todb, Corollary 3.12])

$$(4.7.6) \quad \phi_{ij}: U_j^{\text{nc}}|_{U_{ij}} \xrightarrow{\cong} U_i^{\text{nc}}|_{U_{ij}}, \quad g_{ij}: \phi_{ij}^* \mathcal{V}_{U_i}^{\text{nc}}|_{U_{ij}} \xrightarrow{\cong} \mathcal{V}_{U_j}^{\text{nc}}|_{U_{ij}}$$

such that $\phi_{ij}^{ab} = \text{id}$ and g_{ij}^{ab} is the gluing isomorphism of the universal object \mathcal{V} in (4.3.4). Since the dg-algebra (4.7.1) is determined by the algebra $\Gamma(\mathcal{O}_U^{\text{nc}})$ together with the $Q_{[p,q]}$ -representation $\mathcal{V}_U^{\text{nc}}$ over U^{nc} , the isomorphisms (4.7.6) induce the isomorphisms of NCDG schemes

$$\phi_{ij, \bullet}: (U_{ij}, \mathcal{O}_{U_{ij}, \bullet}^{\text{nc}}|_{U_{ij}}) \xrightarrow{\cong} (U_{ij}, \mathcal{O}_{U_i, \bullet}^{\text{nc}}|_{U_{ij}})$$

giving a quasi NCDG structure on $(N_\gamma, \mathcal{O}_{N_\gamma, \bullet})$. Also under (4.7.4), the isomorphisms $\mathcal{H}_0(\phi_{ij, \bullet})$ give a quasi NC structure on M_γ considered in Theorem 4.7. As a summary, we have obtained the following:

Theorem 4.10. *There exists a smooth quasi NCDG structure on the smooth commutative dg-moduli space $(N_\gamma, \mathcal{O}_{N_\gamma, \bullet})$ whose zero-th truncation gives a quasi NC structure on M_γ in Theorem 4.7.*

5. QUASI NCDG STRUCTURES ON THE MODULI SPACES OF STABLE SHEAVES

In this section, we show that quasi NC structures on the moduli spaces of stable sheaves on projective schemes constructed in [Todb] are obtained as the zero-th truncations of smooth quasi NCDG structures on smooth commutative dg-moduli spaces of stable sheaves. Throughout this section, we assume that $(X, \mathcal{O}_X(1))$ is a connected polarized projective scheme over \mathbb{C} .

5.1. Moduli spaces of stable sheaves. For $F \in \text{Coh}(X)$, let $\alpha(F, t)$ be its Hilbert polynomial

$$\alpha(F, t) := \chi(F \otimes \mathcal{O}_X(t))$$

and $\overline{\alpha}(F, t) := \alpha(F, t)/c$ its reduced Hilbert polynomial, where c is the leading coefficient of $\alpha(F, t)$. Recall the (semi)stability on X :

Definition 5.1. *A coherent sheaf F on X is called (semi)stable if it is a pure sheaf, and for any subsheaf $0 \subsetneq F' \subsetneq F$, we have*

$$(5.1.1) \quad \overline{\alpha}(F', k) < (\leq) \overline{\alpha}(F, k), \quad k \gg 0.$$

Let us take a polynomial $\alpha \in \mathbb{Q}[t]$, which is a Hilbert polynomial of some coherent sheaf on X . Let

$$(5.1.2) \quad \mathcal{M}_\alpha: \text{Sch}/\mathbb{C} \rightarrow \text{Set}$$

be the functor defined by

$$\mathcal{M}_\alpha(T) := \left\{ \mathcal{F} \in \text{Coh}(X \times T) : \begin{array}{l} \mathcal{F} \text{ is } T\text{-flat, } \mathcal{F}_t \text{ for any } t \in T \text{ is} \\ \text{stable with Hilbert polynomial } \alpha \end{array} \right\} / (\text{equiv}).$$

Here \mathcal{F} and \mathcal{F}' are *equivalent* if there is a line bundle \mathcal{L} on T such that $\mathcal{F} \cong \mathcal{F}' \otimes p_T^* \mathcal{L}$, where $p_T: X \times T \rightarrow T$ is the projection. The moduli functor (5.1.2) is not always representable by a scheme, but if we assume that

$$(5.1.3) \quad \text{g.c.d.}\{\alpha(m) : m \in \mathbb{Z}\} = 1$$

then (5.1.2) is represented by a projective scheme M_α (cf. [Muk87]), i.e. there is an isomorphism of functors

$$(5.1.4) \quad \text{Hom}(-, M_\alpha) \xrightarrow{\cong} \mathcal{M}_\alpha.$$

We call α satisfying the condition (5.1.3) as *primitive*. Below, we always assume that α is primitive. Note that the isomorphism (5.1.4) is induced by a universal family

$$\mathcal{U} \in \text{Coh}(X \times M_\alpha).$$

5.2. DG moduli spaces of stable sheaves. Note that $X = \text{Proj}(A)$ for the graded algebra

$$A = \bigoplus_{i \geq 0} H^0(X, \mathcal{O}_X(i)).$$

We use the quiver with relation $(Q_{[p,q]}, I)$ constructed in Definition 4.1 from the above graded algebra A . For $q > p > 0$, we set

$$(5.2.1) \quad \Gamma_{[p,q]}(\mathcal{U}) := \bigoplus_{i=p}^q p_{M*}(\mathcal{U} \otimes p_X^* \mathcal{O}_X(i)).$$

Here p_M, p_X are the projections from $X \times M_\alpha$ to M_α, X respectively. If we take $q \gg p \gg 0$, then (5.2.1) is a flat representation of $(Q_{[p,q]}, I)$ over M_α with the primitive dimension vector

$$(5.2.2) \quad \gamma = (\alpha(p), \alpha(p+1), \dots, \alpha(q)).$$

The object (5.2.1) defines the morphism of schemes

$$(5.2.3) \quad \Upsilon: M_\alpha \rightarrow M_\gamma$$

which is an open immersion by [BFHR14, Corollary 3.4]. Let $M_{[p,q]} \subset M_\gamma$ be the image of Υ . Since both sides of (5.2.3) are projective, the image $M_{[p,q]}$ is a union of connected components of M_γ . We have the isomorphism of schemes

$$(5.2.4) \quad \Upsilon: M_\alpha \xrightarrow{\cong} M_{[p,q]}.$$

In particular, the object (5.2.1) is a universal $(Q_{[p,q]}, I)$ -representation restricted to $M_{[p,q]}$. By replacing \mathcal{U} by $\mathcal{U} \otimes p_M^* \mathcal{L}$ for some line bundle \mathcal{L} on M_α , we may assume that the universal family \mathcal{V}_\bullet given in (4.3.4) restricted to $M_{[p,q]}$ coincides with (5.2.1).

Recall that M_γ is obtained as the zero-th truncation of the smooth commutative dg-scheme $(N_\gamma, \mathcal{O}_{N_\gamma, \bullet})$ given by (4.3.6). By taking a suitable open subset $N_\alpha \subset N_\gamma$ containing $M_{[p,q]}$, the following result was proved in [BFHR14]:

Theorem 5.2. ([BFHR14]) *There is an open subset $N_\alpha \subset N_\gamma$ such that the smooth commutative dg-scheme*

$$(5.2.5) \quad (N_\alpha, \mathcal{O}_{N_\alpha, \bullet} := \mathcal{O}_{N_\gamma, \bullet}|_{N_\alpha})$$

satisfies the following:

- The zero-th truncation of (5.2.5) is isomorphic to M_α .
- For any $[E] \in M_\alpha$, the tangent complex of (5.2.5) at $[E]$ is

$$(5.2.6) \quad \text{Cone}(\mathbb{C} \rightarrow \mathbf{R} \text{Hom}(E, E))[1].$$

5.3. Existence of a quasi NCDG structure. One can also extend the isomorphism (5.2.4) to their NC thickenings. Let

$$h_\alpha: \mathcal{N} \rightarrow \text{Set}$$

be the functor sending $R \in \mathcal{N}$ to the isomorphism classes of triples (f, \mathcal{F}, ψ) :

- f is a morphism of schemes $f: \text{Spec } R^{ab} \rightarrow M_\alpha$.
- \mathcal{F} is an object of $\text{Coh}(X_R)$ which is flat over R , where $X_R := X \times \text{Spf } R$.
- ψ is an isomorphism $\psi: \mathcal{F}^{ab} \xrightarrow{\cong} f^* \mathcal{U}$.

An isomorphism $(f, \mathcal{F}, \psi) \rightarrow (f', \mathcal{F}', \psi')$ exists if $f = f'$, and there is an isomorphism $\mathcal{F} \rightarrow \mathcal{F}'$ in $\text{Coh}(X_R)$ commuting ψ, ψ' .

Proposition 5.3. ([Todb, Proposition 4.13]) *The isomorphism (5.2.4) extends to the isomorphism of functors*

$$\Gamma_{[p,q]}: h_\alpha \xrightarrow{\cong} h_\gamma|_{M_{[p,q]}}.$$

By combining Theorem 4.10, Theorem 5.2 and Proposition 5.3, we obtain the following:

Theorem 5.4. *There is a smooth quasi NCDG structure $\{(U_i, \mathcal{O}_{U_i, \bullet}^{\text{nc}})\}_{i \in \mathbb{I}}$ on the smooth commutative dg-moduli space $(N_\alpha, \mathcal{O}_{N_\alpha, \bullet})$ such that the zero-th truncations*

$$(5.3.1) \quad \{(V_i, \mathcal{O}_{V_i}^{\text{nc}})\}_{i \in \mathbb{I}} := \{\tau_0(U_i, \mathcal{O}_{U_i, \bullet}^{\text{nc}})\}_{i \in \mathbb{I}}$$

is a quasi NC structure on M_α which fit into NC hulls $h_{V_i^{\text{nc}}} \rightarrow h_\alpha|_{V_i}$.

The quasi NC structure in (5.3.1) is the one constructed in [Todb]. By [Todb, Theorem 1.2], it satisfies the following condition. For $[E] \in V_i$, let $\widehat{\mathcal{O}}_{V_i, [E]}^{\text{nc}}$ be the completion of $\mathcal{O}_{V_i}^{\text{nc}}$ at $[E]$. Then it coincides with the pro-representable hull of the NC deformation functor of E developed in [Lau02], [Eri10], [Seg08], [ELO09], [ELO10], [ELO11]. This implies that we have an isomorphism of algebras

$$\widehat{\mathcal{O}}_{V_i, [E]}^{\text{nc}} \cong R_E^{\text{nc}}$$

where R_E^{nc} is the algebra (1.2.2) constructed by the A_∞ -structure (1.2.1).

5.4. An example. We take $X = \mathbb{P}^2$ and α to be the constant function 1. Note that a stable sheaf on X has Hilbert polynomial 1 if and only if it is a skyscraper sheaf \mathcal{O}_x for $x \in \mathbb{P}^2$. Then the moduli space M_α is isomorphic to \mathbb{P}^2 itself. On the other hand, by Beilinson's theorem [Bei78], we have the derived equivalence

$$\mathbf{R} \text{Hom}(\mathcal{E}, -): D^b(\text{Coh}(\mathbb{P}^2)) \xrightarrow{\sim} D^b(\text{mod } A)$$

where \mathcal{E} and A are given by

$$\mathcal{E} = \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-1) \oplus \mathcal{O}_{\mathbb{P}^2}(-2), \quad A = \text{End}(\mathcal{E}).$$

By the above equivalence, one can take $p = 0$ and $q = 2$ in the argument of the previous subsection. The quiver $Q_{[0,2]}$ is described as

$$(5.4.1) \quad \begin{array}{ccccc} & & \xrightarrow{z_{33}} & & \\ & & \xrightarrow{z_{22}} & & \\ & & \xrightarrow{z_{11}} & & \\ 0 & \xrightarrow{x_1} & 1 & \xrightarrow{y_1} & 2 \\ & \xrightarrow{x_2} & & \xrightarrow{y_2} & \\ & \xrightarrow{x_3} & & \xrightarrow{y_3} & \\ & & \xrightarrow{z_{12}} & & \\ & & \xrightarrow{z_{13}} & & \\ & & \xrightarrow{z_{23}} & & \end{array}$$

with relations given by

$$z_{ij} = y_j x_i = y_i x_j, \quad 1 \leq i \leq j \leq 3.$$

The dimension vector (5.2.2) is $\gamma = (1, 1, 1)$. The moduli space N_γ of representations of $Q_{[0,2]}$ without relation is the quotient of the stable locus of $\mathbb{C}^3 \times \mathbb{C}^3 \times \mathbb{C}^6$ by $(\mathbb{C}^*)^{\times 2}$. It contains an open subset $U \subset N_\gamma$ which parametrizes representations of (5.4.1) with $x_3 = y_3 = 1$ and $x_1, x_2, y_1, y_2, z_{ij} \in \mathbb{C}$, i.e.

$$U = \text{Spec } \mathbb{C}[x_1, x_2, y_1, y_2, z_{ij} : 1 \leq i \leq j \leq 3].$$

An NC smooth thickening of U is given by

$$U^{\text{nc}} = \text{Spf } R, \quad R = \mathbb{C}\langle x_1, x_2, y_1, y_2, z_{ij} : 1 \leq i \leq j \leq 3 \rangle_{[ab]}.$$

Let \mathcal{V} be the universal representation of $Q_{[0,2]}$ on N_γ , which is a rank three vector bundle. A lift of $\mathcal{V}|_U$ to U^{nc} is given by the following representation

$$\begin{array}{ccccc} & & \xrightarrow{\cdot z_{33}} & & \\ & & \xrightarrow{\cdot z_{22}} & & \\ & & \xrightarrow{\cdot z_{11}} & & \\ R & \xrightarrow{\cdot x_1} & R & \xrightarrow{\cdot y_1} & R \\ & \xrightarrow{\cdot x_2} & & \xrightarrow{\cdot y_2} & \\ & \xrightarrow{1} & & \xrightarrow{1} & \\ & & \xrightarrow{\cdot z_{12}} & & \\ & & \xrightarrow{\cdot z_{13}} & & \\ & & \xrightarrow{\cdot z_{23}} & & \end{array}$$

The algebra (4.7.2) is then given by

$$\Lambda_{U, \bullet}^{\text{nc}} = R * T(\mathfrak{m}_1^{\otimes 2})$$

where $\mathfrak{m}_1 = H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1))$ and $\mathfrak{m}_1^{\otimes 2}$ is located in degree -1 . It is written as a suitable NC completion of

$$\mathbb{C}\langle x_1, x_2, y_1, y_2, z_{ij}, w_{kl} : 1 \leq i \leq j \leq 3, 1 \leq k, l \leq 3 \rangle$$

where $\deg x_i = \deg y_i = \deg z_{ij} = 0$ and $\deg w_{kl} = -1$ and the differential is given by

$$Q: w_{kl} \mapsto z_{kl} - y_k x_l$$

where we set $z_{kl} = z_{lk}$ if $k > l$ and $x_3 = y_3 = 1$. Then $\text{Spf } \Lambda_{U, \bullet}^{\text{nc}}$ is a smooth affine NCDG structure on its abelization $(U, \mathcal{O}_U \otimes S(\mathfrak{m}_1^{\otimes 2}))$. The zero-th truncation of $\text{Spf } \Lambda_{U, \bullet}^{\text{nc}}$ gives $\mathbb{C}^2 \subset \mathbb{P}^2 = M_{\alpha \equiv 1}$.

5.5. NC virtual structure sheaves on moduli spaces of stable sheaves.

Now we assume that the smooth commutative dg-scheme (5.2.5) is a $[0, 1]$ -manifold, which means that the tangent complex of (5.2.5) has amplitude in $[0, 1]$. By (5.2.6), this is equivalent to the condition

$$(5.5.1) \quad \mathrm{Ext}^i(E, E) = 0, \quad i \geq 3$$

for any $[E] \in M_\alpha$. Applying Definition 2.14 to the quasi NCDG structure in Theorem 5.4, we obtain the d -th NC virtual structure sheaf

$$(\mathcal{O}_{M_\alpha}^{\mathrm{ncvir}})^{\leq d} \in K_0(M_\alpha).$$

By Theorem 3.9 and (5.2.6), we have the following:

Corollary 5.5. *The d -th NC virtual structure sheaf associated to the quasi NCDG structure in Theorem 5.4 is written as*

$$(5.5.2) \quad (\mathcal{O}_{M_\alpha}^{\mathrm{ncvir}})^{\leq d} = \mathcal{O}_{M_\alpha}^{\mathrm{vir}} \otimes_{\mathcal{O}_{M_\alpha}} [S_{\mathcal{O}_{M_\alpha}} L_{\mathcal{O}_{M_\alpha}}^+(\mathcal{E}_\bullet)^{\leq d}].$$

Here $\mathcal{E}_\bullet \rightarrow \tau_{\geq -1} \mathbf{L}_{M_\alpha}$ is a perfect obstruction theory on M_α such that for any $[E] \in M_\alpha$ we have

$$\mathcal{H}_0(\mathcal{E}_\bullet^\vee|_{[E]}) = \mathrm{Ext}^1(E, E), \quad \mathcal{H}_1(\mathcal{E}_\bullet^\vee|_{[E]}) = \mathrm{Ext}^2(E, E).$$

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